1. [APPM 2360 Exam (20 pts)] Let \( \mathbf{u} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \)

(a) (6 pts) Show that \( \mathbf{u}^T \mathbf{u} = \mathbf{I}_1 \), where \( \mathbf{I}_1 \) is the 1 \( \times \) 1 identity matrix.

(b) (6 pts) Compute \( \mathbf{H} = \mathbf{I}_3 - 2 \mathbf{u} \mathbf{u}^T \)

(c) (6 pts) Show that \( \mathbf{H} \) is nonsingular (invertible).

(d) (2 pts) Can Cramer’s Rule be used to solve the system \( \mathbf{H} \mathbf{x} = \mathbf{b} \)? Explain why or why not.

2. [APPM 2360 Exam (15 pts)] Consider the linear system

\[
\begin{align*}
  x - 4y &= 17 \\
  3x - 12y &= k \\
 -2x + 8y &= -34
\end{align*}
\]

(a) (8 pts) Use Gauss-Jordan Reduction to determine the value of \( k \) that makes the system consistent.

(b) (7 pts) Using the value of \( k \) found in part (a), write the solution to the system using the Nonhomogenous Principle \( \mathbf{x} = \mathbf{x}_h + \mathbf{x}_p \).

3. [APPM 2360 Exam (20 pts)] Use the matrix inverse to find the solution to the system

\[
\begin{align*}
  x_1 + 2x_2 + 3x_3 &= 3 \\
  x_2 + 2x_3 &= 1 \\
 -2x_1 + 3x_3 &= 4
\end{align*}
\]

4. [APPM 2360 (25 pts)] The following problems are not related.

(a) (12 pts) Suppose \( \mathbf{A} \), \( \mathbf{B} \), \( \mathbf{C} \) are \( n \times n \) matrices with \( |\mathbf{A}| = 3 \), \( |\mathbf{B}| = 1 \), \( |\mathbf{C}| = 0 \). Calculate the following or explain why they fail to exist.

i. \( |\mathbf{AB}| \)

ii. \( |\mathbf{B}^T| \)

iii. \( |\mathbf{B}^2\mathbf{A}\mathbf{C}^{-1}| \)

iv. \( |\mathbf{D}| \), where \( \mathbf{D} \) is the matrix obtained by interchanging the second and \( n^{th} \) rows of \( \mathbf{A} \)

(b) (9 pts) Let \( \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix} \) and \( \mathbf{B} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \). Calculate the following or explain why they fail to exist.

i. \( \mathbf{A}^T \)

ii. \( \mathbf{A}^T + \mathbf{B} \)

iii. \( (\mathbf{B}^T)^{-1} \)

(c) (4 pts) Let \( \mathbf{A} \) be an \( n \times n \) invertible matrix satisfying \( \mathbf{A}^3 + 2\mathbf{A} = \mathbf{I} \). Find an expression for \( \mathbf{A}^{-1} \).

5. [APPM 2360 Exam (20 pts)] The following problems are not related.

(a) (8 pts) Decide if the following subsets \( \mathcal{W} \) of the given vector space \( \mathcal{V} \) are subspaces. Assume that the standard operations of vector addition and scalar multiplication apply. Justify the correct answer completely for full credit. A simple yes/no will result in zero points.

i. \( \mathcal{V} = \mathcal{C}([0, 1]) ; \quad \mathcal{W} = \left\{ f(t) \left| \int_0^1 f(t) \, dt = 2 \right. \right\} \)

ii. \( \mathcal{V} = \mathcal{M}_{2 \times 3} ; \quad \mathcal{W} = \text{matrices of the form} \begin{bmatrix} 0 & a & b \\ c & 0 & d \end{bmatrix} \text{ where} a, b, c, d \text{ are real numbers.} \)

(b) (4 pts) Determine whether or not the set \( \mathcal{S} = \{ 2, 1 - t, t + t^3 \} \) forms a basis for some vector space. If so, what is its dimension? If not, explain why not.

(c) (8 pts) Find the eigenvalues and eigenvectors of \( \mathbf{A} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \).