1. [APPM 2360 Exam (30 pts)] The following problems are not related.

   (a) (10 pts) Solve the initial value problem $y' = (\sin t)\sqrt{y},\ y(0) = 1$. Write your answer as an explicit function, that is, $y(t) = \cdots$.

   (b) Consider the differential equation $ty' + y = t\sin t$.

      i. (5 pts) Show that $y_h(t) = \dfrac{C}{t}$ is a solution of the associated homogeneous equation. C is an arbitrary constant.

      ii. (10 pts) Use the Euler-Lagrange two-stage method to find a particular solution to the nonhomogeneous equation.

      iii. (5 pts) Use the Nonhomogeneous Principle to write the solution to the nonhomogeneous differential equation.

SOLUTION:

(a) Use separation of variables.

$$\int \frac{\mathrm{d}y}{\sqrt{y}} = \int \sin t\,\mathrm{d}t$$

$$2\sqrt{y} = -\cos t + C \qquad \text{apply initial condition}$$

$$2\sqrt{1} = -\cos 0 + C \implies C = 3$$

$$y(t) = \left(\frac{3}{2} - \frac{1}{2}\cos t\right)^2$$

(b) i. Substitute into the DE.

$$ty_h' + y_h = -tCt^{-2} + Ct^{-1} = -Ct^{-1} + Ct^{-1} = 0$$

ii. Let $y_p = v(t)t^{-1}$. Then $y_p' = -vt^{-2} + v't^{-1}$. Rewrite the DE as $y' + \dfrac{y}{t} = \sin t$ and substitute y_p into this to get

$$-vt^{-2} + v't^{-1} + vt^{-2} = \sin t \implies v' = t\sin t$$

Integration by parts yields $v(t) = -t\cos t + \sin t$ so that $y_p(t) = \dfrac{\sin t}{t} - \cos t$.

iii.

$$y(t) = y_h(t) + y_p(t) = \frac{C}{t} + \frac{\sin t}{t} - \cos t$$

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2. [APPM 2360 Exam (20 pts)] You are making a secret marinade sauce for meat that involves dissolving 100 grams of Special Spice #1 in 10 gallons of vinegar in a large tank. A malefactor has decided to sabotage the mixture by creating a machine that pours 1 gallon, containing 30 grams of Special Spice #1, into the tank every minute. As soon as the machine is turned on, the malefactor creates a hole in the tank that drains the combined mixture from the tank at 2 gallons per minute.

   (a) (10 pts) Set up the initial value problem (IVP) describing this situation. Let $t = 0$ be the moment that the hole is created in the tank.

   (b) (10 pts) Solve the IVP using the integrating factor method. Minimal credit, if any, will be awarded for simply using a formula that yields the result. Instead, show all the steps needed to arrive at the solution.

SOLUTION:

(a) Let $x(t)$ be the amount of Special Spice #1 in the tank at time t. Then the initial value problem governing this situation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 30 - \frac{2x}{10-t},\ x(0) = 100$$

Note that this is valid on the interval $0 \le t < 10$ since the tank is empty after 10 minutes.

(b) Rearranging the ODE to

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2x}{10-t} = 30$$

gives an integrating factor of $\mu(t) = \dfrac{1}{(10-t)^2}$. Multiplying by the integrating factor yields

$$\left[\frac{x}{(10-t)^2}\right]' = \frac{30}{(10-t)^2}$$

which, after integration, gives

$$x(t) = (10-t)^2\left(\frac{30}{10-t} + C\right)$$

Application of the initial condition gives $C = -2$ so that the solution to the initial value problem is

$$x(t) = (10-t)^2\left(\frac{30}{10-t} - 2\right)$$

3. [APPM 2360 Exam (24 pts)] After discovering the culinary sabotage noted in the previous problem, you decide to make a new batch of marinade with your trademark 100 grams of Special Spice #1. In addition to being irresistibly delicious, Special Spice #1 is also an unstable radioactive material with a half-life of 10 days.

   (a) (8 pts) Set up the corresponding initial value problem for this decay problem assuming that the new marinade is made at $t = 0$.

   (b) (8 pts) How much Special Spice #1 will be left after t days?

   (c) (8 pts) Your marinade will lose its flavor if under 10 grams of Special Spice #1 are left. How long after making a fresh batch of the marinade do have to use it?

SOLUTION:

(a) Let y be the amount of Special Spice #1. An exponential decay problem has the form $\mathrm{d}y/\mathrm{d}t = -ky, k > 0$ and initial condition $y(0) = y_0$. Since we are told the half-life is 10 days, $k = \ln 2/10$. Thus the initial value problem is

$$y' = -\frac{\ln 2}{10}y,\ y(0) = y_0$$

(b) The solution of the DE is, by separation of variables,

$$y(t) = 100e^{-(\ln 2/10)t}$$

(c) We need to determine when $y(t) = 10$.

$$10 = 100e^{(-\ln 2/10)t} \implies t = \frac{10\ln 10}{\ln 2}$$

4. [APPM 2360 (16 pts)] The following problems are not related.

   (a) Consider the differential equation $y' = y^3 - 3y^2 - y + 3$.

      i. (5 pts) Find all equilibrium solutions and their stability.

      ii. (5 pts) Plot the phase line for the differential equation.

   (b) (6 pts) Given the differential equation $y' + y = t^2$, draw the isoclines corresponding to slopes of $1, 0, -1$. Be sure to include the line segments showing the slope on each isocline.

SOLUTION:

(a) We can write the ODE as $y' = (y+1)(y-1)(y-3)$ showing that the equilibrium solutions are $y = -1, y = 1, y = 3$.

   i.

$$\begin{aligned} y > 3: &\quad y' > 0 \\ 1 < y < 3: &\quad y' < 0 \\ -1 < y < 1: &\quad y' > 0 \\ y < -1: &\quad y' < 0 \end{aligned}$$

   Thus $y = -1$ and $y = 3$ are unstable and $y = 1$ is stable.

   ii. Phase line.

$\uparrow$

3 ○

$\downarrow$

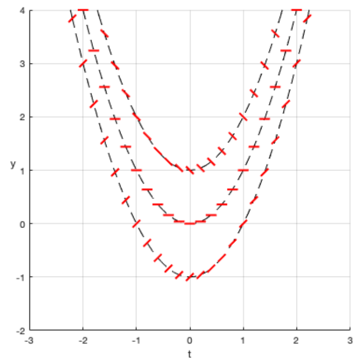
1 ●

$\uparrow$

-1 ○

$\downarrow$

(b) Isoclines are the parabolas $y = t^2 - k$ with $k = -1, 0, 1$.



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5. [APPM 2360 Exam (10 pts)] The following problems are not related.

(a) (5 pts) With a step size of $h = 0.5$, use Euler's method to approximate the solution of the IVP $y' = 2t + y,\ y(1) = 2$ at $t = 2$.

(b) (5 pts) What conclusions can be drawn from Picard's Theorem regarding the existence and unique of solutions to the initial value problem $y' = \sqrt{t+y},\ y(2) = 0$? Briefly explain.

SOLUTION:

(a) Using $y_{n+1} = y_n + h(2t_n + y_n),\ n = 0, 1$ we have

$$y(1.5) \approx y_1 = y_0 + \tfrac{1}{2}\left(2t_0 + y_0\right) = 2 + \tfrac{1}{2}\left[2(1) + 2\right] = 4$$

$$y(2.0) \approx y_2 = y_1 + \tfrac{1}{2}\left(2t_1 + y_1\right) = 4 + \tfrac{1}{2}\left[2\left(\tfrac{3}{2}\right) + 4\right] = \tfrac{15}{2}$$

(b) We have $f(t,y) = \sqrt{t+y}$ which is continuous for $t + y \geq 0$ and $f_y = \dfrac{1}{2\sqrt{t+y}}$ which is continuous if $t + y > 0$. Since the initial point $(2, 0)$ satisfies both inequalities, we can can find a rectangle containing the initial point $(2, 0)$ throughout which both $f(t,y)$ and $f_y(t,y)$ are continuous, satisfying the hypotheses of Picard's Theorem. The theorem guarantees the existence of a unique solution to the IVP.

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