1. (60 pts) The following problems are unrelated.

   (a) Find the general solution to the linear system of differential equations $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{\mathbf{x}}$.

   (b) Compute the Laplace transform of the function $f(t) = \begin{cases} 0 & t < 0 \\ \sin(t) & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$.

   (c) Determine the inverse Laplace transform of the following:

   i. $F(s) = \dfrac{1}{s^2 + 2s + 4}$ ii. $G(s) = \dfrac{3s + 11}{s^2 - s - 6}$

   (d) For which initial conditions can we guarantee a unique solution exists to the initial value problem

$$y' = \frac{1}{y - t}, \quad y(t_0) = y_0$$

   (e) i. Show that the set of diagonal 3×3 matrices is a vector space.

   ii. Let $\mathbb{W}$ be a subspace of $\mathbb{P}_4$, $\mathbb{W} = \{p(x) \in \mathbb{P}_4 \mid p(x) = ax^4 + bx^2 + c\}$. Determine the dimension and a basis of the subspace $\mathbb{W}$.

   (f) For which values of b does the system $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix} \vec{\mathbf{x}}$ have a solution that is a spiral toward the origin?

Solution:

(a) The coefficient matrix has the eigenvalues

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3 - \lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \implies \lambda_1 = -1, \lambda_2 = -2$$

with corresponding eigenvectors

$$\lambda_1 = -1 : \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \implies v_1 + v_2 = 0 \implies v_1 = -v_2 \implies \vec{\mathbf{v}}_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -2 : \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \implies 2v_1 + v_2 = 0 \implies v_1 = -\frac{1}{2}v_2 \implies \vec{\mathbf{v}}_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The general solution is then given by

$$\boxed{\vec{\mathbf{x}} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

(b) The piecewise function can be rewritten as

$$f(t) = \sin(t)\,(\text{step}(t) - \text{step}(t-\pi))$$

So the Laplace transform is given by

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \mathscr{L}\{\sin(t)\text{step}(t)\} - \mathscr{L}\{\sin(t)\text{step}(t-\pi)\}\\ &= \mathscr{L}\{\sin(t)\} - e^{-\pi s}\mathscr{L}\{\sin(t+\pi)\}\\ &= \frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}\{\sin(t)\cos(\pi) + \sin(\pi)\cos(t)\}\\ &= \frac{1}{s^2+1} + e^{-\pi s}\mathscr{L}\{\sin(t)\} = \frac{1+e^{-\pi s}}{s^2+1}\end{aligned}$$

(c) i. Complete square in the denominator and then manipulate

$$F(s) = \frac{1}{s^2+2s+1-1+4} = \frac{1}{(s+1)^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}$$

which is of an appropriate form to compute the inverse Laplace transform using the table provided. The function in the time domain is then

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}\right\} = \boxed{\frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)}$$

ii. Factoring the denominator, the function can be rewritten as

$$G(s) = \frac{3s+11}{(s-3)(s+2)},$$

Seeking a partial fraction decomposition of the form $\frac{A}{s-3} + \frac{B}{s+2}$, we find that A and B are given by solutions to the linear system

$$\begin{aligned} A + B &= 3\\ 2A - 3B &= 11\end{aligned}$$

so that $A = 4$ and $B = -1$. Therefore

$$G(s) = \frac{4}{s-3} - \frac{1}{s+2},$$

and the inverse Laplace transform is

$$g(t) = \mathscr{L}^{-1}\left\{\frac{4}{s-3} - \frac{1}{s+2}\right\} = \boxed{4e^{3t} - e^{-2t}}$$

(d) Here we apply both parts of Picard's theorem with the function $f(t,y) = \frac{1}{y-t}$. Since $f(t,y)$ is continuous for all values such that $y \neq t$, then we can guarantee a solution to the IVP exists for $y_0 \neq t_0$. Similarly, $f_y(t,y) = \frac{1}{(y-t)^2}$ is continuous for all values such that $y \neq t$, guaranteeing there is a unique solution to the IVP so long as $y_0 \neq t_0$.

(e) i. The space of 3×3 matrices contains the zero matrix, so the space is clearly nonempty. Now we need to show the closure properties. Let

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & 0 & 0 \\ 0 & a_{22} + b_{22} & 0 \\ 0 & 0 & a_{33} + b_{33} \end{bmatrix},$$

which is a diagonal 3×3 matrix, so the set is closed under addition. Similarly, for $c \in \mathbb{R}$

$$cA = \begin{bmatrix} ca_{11} & 0 & 0 \\ 0 & ca_{22} & 0 \\ 0 & 0 & ca_{33} \end{bmatrix},$$

is a 3×3 diagonal matrix so that the set is closed under scalar multiplication. Therefore, the set of 3×3 diagonal matrices forms a vector space.

ii. A basis for the subspace is $\{x^4, x^2, 1\}$ so that the dimension is 3.

(f) The solution will spiral towards the origin if the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}$ have negative real part and nonzero imaginary part. The eigenvalues are given when

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -b - \lambda \end{vmatrix} = -\lambda(-b - \lambda) + 1 = \lambda^2 + b\lambda + 1 = 0$$

So the eigenvalues of A are

$$\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4}}{2}$$

We will therefore have a spiral solution towards the origin so long as $b > 0$ and $b^2 - 4 < 0$. The latter inequality is equivalent to $-2 < b < 2$. Since b must be positive, this requires $0 < b < 2$.

Alternatively, using the Tr$A/|A|$ parameter plane graph, for an attracting spiral we need TrA negative, $|A|$ positive and $|A| > \frac{1}{4}(\text{Tr}A)^2$ (above the parabola). Tr$A = -b < 0$ requires $b > 0$. To stay above the parabola we need $1 > \frac{1}{4}(-b)^2 \implies 4 > b^2 \implies 2 > b > -2$. Combining these two requirements gives $0 < b < 2$ for an attracting spiral.

2. (25 pts) Solve the initial value problem with Laplace transforms.

$$y' + 2y = \delta(t - 1) - 3, \quad y(0) = 1.$$

Solution: Taking the Laplace transform of both sides, we obtain

$$\begin{aligned} \mathscr{L}\{y'\} + 2\mathscr{L}\{y)\} &= \mathscr{L}\{\delta(t - 1)\} - \mathscr{L}\{3\} \\ \implies s\mathscr{L}\{y\} - y(0) + 2\mathscr{L}\{y\} &= e^{-s} - \frac{3}{s} \\ \implies (s + 2)\, Y(s) &= 1 + e^{-s} - \frac{3}{s} \end{aligned}$$

where $Y(s) = \mathscr{L}\{y\}$. Solving for $Y(s)$, we find

$$\begin{aligned} Y(s) &= \frac{1}{s+2} + \frac{e^{-s}}{s+2} - \frac{3}{s(s+2)} \qquad \text{(partial fractions)} \\ &= \frac{1}{s+2} + \frac{e^{-s}}{s+2} - \frac{3/2}{s} + \frac{3/2}{s+2} \\ &= \frac{5/2}{s+2} + \frac{e^{-s}}{s+2} - \frac{3/2}{s} \end{aligned}$$

Taking the inverse Laplace transform, we find the solution to the initial value problem is

$$y(t) = \mathscr{L}^{-1}\left\{\frac{5/2}{s+2} + \frac{e^{-s}}{s+2} - \frac{3/2}{s}\right\} = \boxed{\frac{5}{2}e^{-2t} + e^{-2(t-1)}\text{step}(t-1) - \frac{3}{2}}$$

3. (25 pts) Consider the first order equation

$$t\frac{dy}{dt} - y = t^2, \quad y(1) = 3.$$

(a) Determine an integrating factor to solve the differential equation.

(b) Use the integrating factor you found in part (a) to find the general solution to the differential equation.

(c) Apply the initial condition to determine the solution to the initial value problem.

(d) Use Euler's method with a step size of $h = 1$ to approximate $y(2)$.

Solution:

(a) First we need to rewrite the ODE so that the leading coefficient is 1.

$$y' - \frac{1}{t}y = t$$

The integrating factor is then $\mu = e^{-\int \frac{1}{t}dt} = \boxed{\frac{1}{t}}$.

(b) Multiplying the equation by the integrating factor, we have

$$\frac{1}{t}y' - \frac{1}{t^2}y = 1 \implies \frac{d}{dt}\left[\frac{1}{t}y\right] = 1.$$

Integrating with respect to t gives

$$\frac{1}{t}y = t + C$$

so that the general solution is

$$\boxed{y = t^2 + Ct}$$

(c) Applying the initial condition $y(1) = 3$ gives

$$\begin{aligned} 3 &= 1 + C \\ \implies C &= 2 \end{aligned}$$

so that the solution to the IVP is $\boxed{y(t) = t^2 + 2t}$.

(d) Euler's method with $h = 1$ gives the approximation at $t = 2$ as

$$y_2 = y_1 + hf(t_1, y_1),$$

where $t_1 = 1$ and $y_1 = 3$, so that $f(t_1, y_1) = t_1 + \frac{y_1}{t_1} = 4$. So the approximation of $y(2)$ from Euler's method with a step size of $h = 1$ is

$$y(2) \approx y_2 = 3 + 1(4) = \boxed{7}$$

4. (20 pts) In this problem we will solve the differential equation

$$y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = 2(1-t)e^{-t}$$

(a) Knowing that $y_1 = t$ and $y_2 = e^t$ are solutions to the homogeneous differential equation, use variation of parameters to determine the particular solution of the differential equation.

(b) Give the general solution to the differential equation.

Solution:

(a) Using variation of parameters, the particular solution is $y_p = v_1 y_1 + v_2 y_2$. The variable coefficients are then

$$v_1 = -\int \frac{y_2 f}{W(y_1, y_2)}, \quad v_2 = \int \frac{y_1 f}{W(y_1, y_2)}$$

where $f = 2(1-t)e^{-t}$ and $W(y_1, y_2) = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix} = (t-1)e^t$. Now, computing v_1 and v_2

$$\begin{aligned} v_1 &= -\int \frac{e^t\left(2(1-t)e^{-t}\right)}{(t-1)e^t}dt \\ &= 2\int e^{-t}dt \\ &= -2e^{-t} \end{aligned}$$

$$\begin{aligned} v_2 &= \int \frac{t\left(2(1-t)e^{-t}\right)}{(t-1)e^t}dt \\ &= -2\int te^{-2t}dt \quad \text{(integration by parts)} \\ &= \frac{1}{2}e^{-2t} + te^{-2t} \end{aligned}$$

Now, since $y_p = v_1 y_1 + v_2 y_2$, we find

$$\begin{aligned} y_p &= -2e^{-t}t + \left(\frac{1}{2}e^{-2t} + te^{-2t}\right)e^t \\ &= \boxed{\frac{1}{2}e^{-t} - te^{-t}} \end{aligned}$$

(b) The general solution of the ODE is then given by

$$\boxed{y(t) = c_1 t + c_2 e^t + \frac{1}{2}e^{-t} - te^{-t}}$$

5. (20 pts) Consider the nonlinear system of equations

$$\frac{dx}{dt} = (2 - x - y),$$
$$\frac{dy}{dt} = -y(1 - x)$$

(a) Compute the nullclines of the system.

(b) Identify the equilibrium solution(s) to the system of ordinary differential equations.

Solution:

(a) The v nullclines (isoclines of vertical slope) occur where $dx/dt = 0$, so that these nullclines are $\boxed{y = 2 - x}$. Similarly the h nullclines (isoclines of horizontal slope) occur where $dy/dt = 0$

$$-y(1 - x) = 0 \implies \boxed{y = 0, \quad x = 1}$$

(b) The equilibrium solutions are found when dx/dt and dy/dt are simultaneously zero. The equilibrium solutions are

$$\boxed{(2, 0), \quad (1, 1)}$$