APPM 2360 **Final Exam** **Summer 2019**

On the front of your Bluebook write: (1) your name, (2) your section number: **300** for Thaler or **301** for Sprenger and (3) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A **two** sided crib sheet is allowed, which you can take with you when you finish the exam. Make sure to read all instructions carefully and $\boxed{\text{box}}$ your final answer.

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1. (60 pts) The following problems are unrelated.

   (a) Find the general solution to the linear system of differential equations $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{\mathbf{x}}$.

   (b) Compute the Laplace transform of the function $f(t) = \begin{cases} 0 & t < 0 \\ \sin(t) & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$.

   (c) Determine the inverse Laplace transform of the following:

   i. $F(s) = \dfrac{1}{s^2 + 2s + 4}$ ii. $G(s) = \dfrac{3s + 11}{s^2 - s - 6}$

   (d) For which initial conditions can we guarantee a unique solution exists to the initial value problem

   $$y' = \frac{1}{y - t}, \quad y(t_0) = y_0$$

   (e) i. Show that the set of diagonal 3×3 matrices is a vector space.

   ii. Let $\mathbb{W}$ be a subspace of $\mathbb{P}_4$, $\mathbb{W} = \{p(x) \in \mathbb{P}_4 \mid p(x) = ax^4 + bx^2 + c\}$. Determine the dimension and a basis of the subspace $\mathbb{W}$.

   (f) For which values of b does the system $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix} \vec{\mathbf{x}}$ have a solution that is a spiral toward the origin?

2. (25 pts) Solve the initial value problem with Laplace transforms.

   $$y' + 2y = \delta(t - 1) - 3, \quad y(0) = 1.$$

3. (25 pts) Consider the first order equation

   $$t\frac{dy}{dt} - y = t^2, \quad y(1) = 3.$$

   (a) Determine an integrating factor to solve the differential equation.

   (b) Use the integrating factor you found in part (a) to find the general solution to the differential equation.

   (c) Apply the initial condition to determine the solution to the initial value problem.

   (d) Use Euler's method with a step size of $h = 1$ to approximate $y(2)$.

$\boxed{\text{There are more problems on the back!}}$

4. (20 pts) In this problem we will solve the differential equation

$$y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = 2(1-t)e^{-t}$$

(a) Knowing that $y_1 = t$ and $y_2 = e^t$ are solutions to the homogeneous differential equation, use variation of parameters to determine the particular solution of the differential equation.

(b) Give the general solution to the differential equation.

5. (20 pts) Consider the nonlinear system of equations

$$\frac{dx}{dt} = (2 - x - y),$$
$$\frac{dy}{dt} = -y(1 - x)$$

(a) Compute the nullclines of the system.

(b) Identify the equilibrium solution(s) to the system of ordinary differential equations.

TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s)$	s domain	$f(t)$	$F(s)$	s domain
1	$\frac{1}{s}$	$s > 0$	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$ $n > 0$, integer
e^{at}	$\frac{1}{s-a}$	$s > a$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$ $n > 0$, integer
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$	$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a\rvert$	$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a\rvert$
$\delta(t-c)$	e^{-cs}	$c \geq 0, s > 0$	$\text{step}(t-c)$	$\frac{e^{-cs}}{s}$	$c \geq 0, s > 0$
$f'(t)$	$sF(s) - f(0)$	depends on $f(t)$	$f(t-c)\text{step}(t-c)$	$e^{-cs}F(s)$	$c \geq 0, s > 0$

n^{th} order derivative: $\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

POTENTIALLY HELPFUL TRIGONOMETRIC IDENTITIES

$$\cos(u+v) = \cos u\cos v - \sin u\sin v \qquad \sin(u+v) = \sin u\cos v + \cos u\sin v$$
$$\cos(u-v) = \cos u\cos v + \sin u\sin v \qquad \sin(u-v) = \sin u\cos v - \cos u\sin v$$