1. (12 pts) **True/False** (answer True if it is always true otherwise answer False). No justification is required as there is no partial credit on this question.

   (a) The differential equation $\ddot{x} - 2\dot{x} + 4x = 0$ describes a critically damped harmonic oscillator.

   (b) The set $\left\{e^{2t}, e^{-t}, te^{-t}\right\}$ forms a basis for the solution space of $y''' - 3y' - 2y = 0$.

   (c) A second order differential equation always has two linearly independent solutions.

   (d) If y_1 and y_2 are solutions to $ay'' + by' + cy = f(t)$, then $y = y_1 - y_2$ is a solution to $ay'' + by' + cy = 0$.

   Solution:

   (a) False

   (b) True

   (c) False

   (d) True

2. (20 pts) The following questions are unrelated. Answer each question and justify your response for full credit.

   (a) Determine the basis for the solution space of the differential equation $y^{(4)} - y = 0$.

   (b) For which values of ω_f does the system $\ddot{x} + 4x = \cos(\omega_f t)$ exhibit resonance?

   (c) Determine the steady state solution to the forced mass spring system

   $$\ddot{x} + 2\dot{x} + 3x = 4\cos(t)$$

   (d) For the following, determine the form of the particular solution so that we can apply the method of undetermined coefficients. If the method of undetermined coefficients can not be used, clearly state so.

      i. $y'' + 5y = e^{-t}$

      ii. $y'' - 4y' - 12y = t^{-2}$

      iii. $y'' + y = t\cos t$

   Solution:

   (a) We seek a solution of the form e^{rt}, so that we have the characteristic equation

   $$r^4 - 1 = (r^2+1)(r^2-1) = (r^2+1)(r+1)(r-1) = 0 \implies r = \pm 1, \pm i$$

   The general solution is then given by

   $$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

   The basis for the solution space is

   $$\left\{e^t, e^{-t}, \cos t, \sin t\right\}$$

(b) The natural frequency of the system is $\omega_0 = 2$, so the system experiences resonance when $\omega_0 = \omega_f = 2$.

(c) Since the system is a forced mass spring system, the steady state solution is just the particular solution. We can compute it using the formulas in the book (page 268), or from undetermined coefficients. The steady state solution is then

$$x_{\text{ss}}(t) = \sin(t) + \cos(t)$$

(d) i. $y_p = Ae^{-t}$

ii. Method of undetermined coefficient can not be used.

iii. $y_p = (At^2 + Bt)\cos(t) + (Ct^2 + Dt)\sin(t)$

3. (24 pts) An object with mass 2 kg is attached to a spring with spring constant $k = 1$ N/m. The unforced motion is damped with a damping constant of 2 N/m/s. The position of the mass at any time t will be denoted by $x(t)$.

(a) Write a differential equation for the position of the mass, $x(t)$.

(b) Solve the differential equation you found in part (a). You must show your work to receive full credit.

(c) If the initial displacement of the mass is $x(0) = 2$ and the initial velocity is $x'(0) = 0$, what is the motion of the mass, $x(t)$, corresponding to this initial value problem?

(d) Sketch the solution determined in part (c).

(e) Characterize the motion of the oscillator as underdamped, overdamped, or critically damped.

Solution:

(a) The ODE describing the position of the mass is

$$2\ddot{x} + 2\dot{x} + x = 0$$

(b) Seeking a solution of the form $x(t) = e^{rt}$, we find

$$\begin{aligned} 2r^2 + 2r + 1 &= 0, \\ \implies r &= \frac{-2 \pm \sqrt{4 - 4(2)}}{4} \\ \implies r &= \frac{-2 \pm \sqrt{-4}}{4} \\ \implies r &= -\frac{1}{2} \pm \frac{1}{2}i \end{aligned}$$

The general solution is then

$$x(t) = e^{-\frac{1}{2}t}\left[c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right)\right]$$

(c) For this portion, we will need to compute $\dot{x}$, given by

$$\dot{x} = \frac{1}{2}e^{-\frac{t}{2}}\left((c_1 - c_2)\cos\left(\frac{t}{2}\right) - (c_1 + c_2)\sin\left(\frac{t}{2}\right)\right).$$

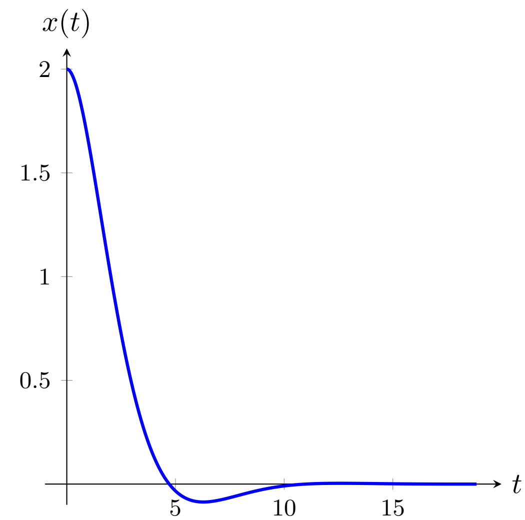
So applying the initial conditions

$$
\begin{aligned}
x(0) = 2 &\implies c_1 = 2 \\
\dot{x}(0) = 0 &\implies \frac{1}{2}c_1 - \frac{1}{2}c_2 = 0 \\
&\implies \boxed{c_1 = c_2 = 2}
\end{aligned}
$$

The solution to the IVP is

$$x(t) = 2e^{-\frac{t}{2}}\left(\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right)$$

(d) The solution should resemble exponential decay with oscillations



(e) The motion of the oscillator is $\boxed{\text{underdamped}}$

4. (20 points) In this problem we will investigate the initial value problem

$$
\begin{aligned}
y'' - y &= 8te^t \\
y(0) &= 1 \\
y'(0) &= 1
\end{aligned}
$$

(a) Find the homogeneous solution of the differential equation.

(b) Determine an appropriate guess for the form of the particular solution.

(c) Use the method of undetermined coefficients to solve for the particular solution using your guess in part (b).

(d) Determine the solution of the initial value problem.

Solution:

(a) The homogeneous solution satisfies

$$y'' - y = 0$$

So the homogeneous solution is

$$y_h(t) = c_1e^t + c_2e^{-t}$$

(b) The initial guess for the particular solution would be

$$y_p = (At + B)e^t,$$

which has a part identical to the homogeneous solution, so we need to modify our particular solution guess to be

$$y_p = (At^2 + Bt)e^t$$

(c) To compute the coefficients, we need to first compute

$$\begin{aligned} y_p' &= (At^2 + (2A + B)t + B)e^t \\ y_p'' &= (At^2 + (4A + B)t + 2A + 2B)e^t \end{aligned}$$

Plug this into the nonhomogeneous problem, we obtain the equations

$$\begin{aligned} 4A &= 8 \\ 2A + 2B &= 0 \end{aligned}$$

so that

$$A = 2 \quad B = -2$$

(d) The general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + (2t^2 - 2t)e^t$$

Applying the initial conditions gives

$$\begin{aligned} y(0) = 1 &\implies c_1 + c_2 = 1 \\ y'(0) = 1 &\implies c_1 - c_2 - 2 = 1 \end{aligned}$$

so that $c_1 = 2$ and $c_2 = -1$ the solution to the IVP is then

$$y(t) = 2e^t - e^{-t} + (2t^2 - 2t)e^t$$

5. (24 points) Consider the differential equation

$$y''' + 2y'' + y' = e^t$$

(a) Use the variable substitution $x = y'$ (so $x' = y''$ and $x'' = y'''$) to write a second order differential equation for x.

(b) Solve for the homogeneous solution, x_h, of the differential equation you found in part (a).

(c) Use the method of *variation of parameters* to determine the particular solution, x_p, to the equation you found in part (a).

(d) Write the general solution to the differential equation in part (a).

(e) Find $y(t)$, the solution of the original differential equation, using the formula $y(t) = \int x(t)\,\mathrm{d}t + C$, where C is a constant.

Solution:

(a) Letting $x = y'$, then x satisfies the ODE

$$x'' + 2x' + x = e^t$$

(b) The homogeneous solution is determined by seeking $x(t) = e^{rt}$ so that

$$r^2 + 2r + 1 = 0 \implies r = -1 \text{ (multiplicity 2).}$$

The homogeneous solution is then

$$x(t) = c_1 e^{-t} + c_2 t e^{-t}$$

(c) Using variation of parameters, we seek a particular solution of the form $x_p = v_1 x_1 + v_2 x_2$ where

$$v_1 = -\int \frac{x_2 f}{W(x_1, x_2)}\,\mathrm{d}t, \quad v_2 = \int \frac{x_1 f}{W(x_1, x_2)}\,\mathrm{d}t$$

Here we choose $x_1 = e^{-t}$ and $x_2 = te^{-t}$, so that $W(x_1, x_2) = e^{-2t}$. Therefore

$$v_1 = -\int t e^{2t}\,\mathrm{d}t, \quad v_2 = \int e^{2t}\,\mathrm{d}t$$

Computing the integrals, we find

$$v_1 = \frac{e^{2t}}{4}\left(1 - 2t\right), \quad v_2 = \frac{e^{2t}}{2}$$

Using these functions to compute the particular solution, we have

$$\begin{aligned} x_p &= v_1 x_1 + v_2 x_2 \\ &= \frac{1}{4}e^t(1 - 2t) + \frac{1}{2}te^t \\ &= \frac{1}{4}e^t \end{aligned}$$

(d) The general solution to the ODE in part (a) is then

$$x = c_1 e^{-t} + c_2 t e^{-t} + \frac{e^t}{4}$$

(e) Using the formula, then

$$y(t) = -c_1 e^{-t} - c_2 e^{-t} - c_2 t e^{-t} + \frac{e^t}{4} + C,$$

If we redefine constants then we find

$$y(t) = \tilde{c}_1 e^{-t} + \tilde{c}_2 t e^{-t} + \tilde{c}_3 + \frac{1}{4}e^t$$