1. (15 pts) **True/False** (answer True if it is always true otherwise answer False). No justification is required as there is no partial credit on this question.

   (a) The vector $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$.

   (b) The dimension of the vector space $\mathbb{P}_3$ is 3.

   (c) if P and D are invertible matrices, then $(PDP^{-1})^{-1} = PD^{-1}P^{-1}$.

   (d) If B is an $n \times n$ matrix with nonzero determinant, then span$\{\text{col } B\} = \mathbb{R}^n$, where col B denotes the column space of B.

   (e) The set of all triangular (upper and lower) 2×2 matrices is a vector subspace of $\mathbb{M}_{2\times 2}$.

   Solution:

   (a) True

   (b) False

   (c) True

   (d) True

   (e) False

2. (20 pts) The following questions are unrelated. Answer each question and justify your response for full credit.

   (a) Find the dimension of the subspace of $\mathbb{P}_3$ spanned by

   $$\left\{ t^3 + t^2,\ t^3 - 2t - 1,\ t^2 + t,\ t + 1 \right\}.$$

   (b) Determine the basis for the space of matrices $A = \begin{bmatrix} 2a & b \\ 3a + b & b \end{bmatrix}$, where $a, b \in \mathbb{R}$.

   (c) Compute A^{-1} for $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

   (d) For the given matrix $A = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix}$, compute the eigenvalues λ_1 and λ_2 and show that $\det(A) = \lambda_1 \lambda_2$.

   Solution:

   (a) Here we can construct a matrix whose elements consist of the coefficients of $t^3, t^2, t, 1$ which is generated by computing $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 + c_4\vec{\mathbf{v}}_4 = 0$.

   $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \vec{\mathbf{c}} = \mathbf{0}$$

Computing the RREF, of the coefficient matrix, we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So only three vectors in the set are linearly independent, so the dimension of the subspace spanned by the functions is $\boxed{3}$

(b) Let us rewrite the matrix A as

$$A = a\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

which gives the basis directly as

$$\left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

(c) Augmenting the matrix with the 3×3 identity matrix and row reducing, we find the inverse to be

$$A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix}$$

(d) We compute the eigenvalues by computing

$$\begin{aligned} &\det(A - \lambda I) = 0 \\ \implies &\begin{vmatrix} -\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) - 4 \\ &\qquad\qquad\qquad = \lambda^2 - 2\lambda - 4 \end{aligned}$$

Using the quadratic formula, we find

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{4+16}}{2} \\ &= 1 \pm \sqrt{5} \end{aligned}$$

So $\boxed{\lambda_1\lambda_2 = (1-\sqrt{5})(1+\sqrt{5}) = -4}$. Similarly, $\det(A) = |A| = -4$, the same result.

3. (25 pts) For the following problem, we consider the linear system of equations

$$\begin{aligned} x - 2y + z &= 0 \\ 2x + y - 3z &= 5 \\ 4x - 7y + z &= -1 \end{aligned}$$

(a) Rewrite the system as an augmented matrix $(A|\vec{\mathbf{b}})$ where A is a matrix and $\vec{\mathbf{b}}$ is a column vector.

(b) Use Gauss-Jordan elimination to write the matrix from part (a) in reduced row echelon form (RREF).

(c) Based on your result in part (b), what is the solution to linear system of equations? Verify this result by plugging in your result to the original system.

(d) Determine the solution to the problem $A\vec{\mathbf{x}} = \mathbf{0}$, where A is the same matrix as in part (a).

Solution:

(a) The augmented matrix corresponding to the linear system is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

(b) Performing Gauss-Jordan elimination we have

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

$$R_2^* = R_2 - 2R_1, \quad R_3^* = R_3 - 4R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$R_2^* = \tfrac{1}{5}R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$R_1^* = R_1 + 2R_2, \quad R_3^* = R_3 - R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$R_3^* = -\tfrac{1}{2}R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$R_1^* = R_1 + R_3, \quad R_2^* = R_2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(c) Based on the result in part (b), the solution to the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(d) Since the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution, then the system $A\vec{\mathbf{x}} = 0$ only has the trivial solution, i.e. $\vec{\mathbf{x}} = 0$

4. (20 pts) In this problem we will consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 7+k & -3 \\ 0 & 4 & k \end{bmatrix}$$

(a) For which values of k is A invertible?

(b) Compute the eigenvalues of A when $k = -2$.

(c) Using the eigenvalues computed in (b), compute the eigenvector(s) corresponding to the *smallest* eigenvalue of A when $k = -2$.

Solution:

(a) The answer is given when $\det A \neq 0$

$$\begin{vmatrix} 1 & 0 & 10 \\ 0 & 7+k & -3 \\ 0 & 4 & k \end{vmatrix} = (7+k)k + 12$$
$$= k^2 + 7k + 12$$
$$= (k+3)(k+4).$$

This shows that so long as $k \neq -3$ or $k \neq -4$, then A is invertible.

(b) When $k = -2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 5 & -3 \\ 0 & 4 & -2 \end{bmatrix},$$

and we compute eigenvalues by computing λ such that $\det(A - \lambda I) = 0$ so we have

$$\begin{vmatrix} 1-\lambda & 0 & 10 \\ 0 & 5-\lambda & -3 \\ 0 & 4 & -2-\lambda \end{vmatrix} = (1-\lambda)\left((5-\lambda)(-2-\lambda)+12\right)$$
$$= (1-\lambda)\left(\lambda^2 - 3\lambda + 2\right)$$
$$= -(\lambda-1)^2(\lambda-2),$$

so the eigenvalues are $\lambda = 1$ with multiplicity 2 and $\lambda = 2$ with multiplicity 1.

(c) We need to compute the eigenvector(s) for $\lambda = 1$. To that end we want to find nonzero vectors $\vec{\mathbf{v}}$ such that

$$(A - I)\vec{\mathbf{v}} = \mathbf{0}$$
$$\implies \begin{bmatrix} 0 & 0 & 10 \\ 0 & 4 & -3 \\ 0 & 4 & -3 \end{bmatrix} \vec{\mathbf{v}} = \mathbf{0} \implies \begin{bmatrix} 0 & 4 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{\mathbf{v}} = \mathbf{0},$$

So that the eigenvector is $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

5. (20 pts) In the following problems, $\mathbb{W}$ is a subset of a vector space $\mathbb{V}$. Determine whether $\mathbb{W}$ is a subspace. If $\mathbb{W}$ is a subspace, prove it. Otherwise provide an explanation or counterexample to show why $\mathbb{W}$ does not form a subspace of $\mathbb{V}$. No credit will be given for responses without justification.

(a) $\mathbb{V} = \mathbb{M}_{2\times 2}$, $\mathbb{W}$ is the set of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 0$.

(b) $\mathbb{V} = \mathcal{C}([0,1])$, $\mathbb{W} = \left\{ f(x) \in \mathcal{C}([0,1]) \;\middle|\; \int_0^1 f(x)dx = 0 \right\}$.

(c) $\mathbb{V} = \mathcal{C}^1(-\infty,\infty)$, $\mathbb{W} = \{f(x) \,|\, f' = f^2\}$.

Solution:

(a) $\mathbb{W}$ forms a subspace. Clearly the set is nonempty. Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ with $a_1 + d_1 = 0$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ with $a_2 + d_2 = 0$ be elements in the set with a, b scalars. Then

$$aA + bB = a \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + b \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} aa_1 + ba_2 & ab_1 + bb_2 \\ ac_1 + bc_2 & ad_1 + bd_2 \end{bmatrix}$$

Now $aa_1 + ba_2 + ad_1 + bd_2 = a(a_1 + d_1) + b(a_2 + d_2) = a(0) + b(0) = 0$ implying that $aA + bB$ is in the set and the set is closed under linear combination and therefore is a vector subspace.

(b) $\mathbb{W}$ forms a subspace. The set is nonempty. Let f and g be elements in the set. Then

$$\int_0^1 f(x)dx = 0 \quad \text{and} \quad \int_0^1 g(x)dx = 0.$$

Let a, b be scalars. Then

$$\int_0^1 (af + bg)dx = a \int_0^1 f(x)dx + b \int_0^1 g(x)dx = a(0) + b(0) = 0$$

implying that $af + bg$ is in the set, further implying that the set is closed under linear combination and therefore a vector subspace.

(c) $\mathbb{W}$ does not form a vector space. Since the set is defined as the solution set of a nonlinear DE, then solutions do not satisfy the scalar multiplication property. More specifically, suppose f_1 is in the subset. Then it solves the DE, $f_1' = f_1^2 \implies f_1' - f_1^2 = 0$. Now consider cf_1. Then $(cf_1)' - (cf_1)^2 = cf_1' - c^2 f_1^2 = c(f_1' - cf_1^2) \neq 0$ necessarily. Thus the set is not closed under scalar multiplication and consequently not a vector subspace. Note that the set is not close under vector addition either.