1. (16 pts) **True/False** (answer True if it is always true otherwise answer False). No justification is required as there is no partial credit on this question.

   (a) The operator $L[y] = y'' + e^t y' + \cos(t)y$ satisfies the two properties of linear operators.

   (b) A population of bacteria doubles in population every 10 days. The population size is described by the differential equation $\frac{dy}{dt} = 10y$ where $t = 1$ corresponds to 1 day.

   (c) Let $y' = f(y)$, where f is continuous. An $f(y)$ exists such that there are only two equilibrium solutions, both of which are stable.

   (d) The equation $y'' + y^2 = 0$ is a linear, second order, homogeneous differential equation.

   Solution:

   (a) True

   (b) False

   (c) False

   (d) False

2. (20 pts) The following questions are unrelated. Answer each question and justify your response for full credit.

   (a) Compute the equilibrium solutions of the differential equation $y' = (1 - y^2)y$ and classify their stability.

   (b) Given the initial value problem

   $$y' = (y - t)^{2/3}, \quad y(t_0) = y_0,$$

   for which initial conditions (t_0, y_0) are we guaranteed that there exists a unique solution to the initial value problem?

   (c) Solve the differential equation

   $$y' = y - y\cos(t)$$

   What is the long term behavior of the solution if $y(0) = \sqrt{2}$?

   Solution:

   (a) The equilibrium solutions occur when $y' = 0$, giving the algebraic system

   $$(y^2 - 1)y = 0,$$

   which has the solutions $y = \pm 1, 0$. The stability can be computed by phase lines, which gives

   - $y = 0$ is unstable
   - $y = \pm 1$ are both stable

(b) The right hand side of the differential equation is $f(t, y) = (y - t)^{2/3}$, which has the partial derivative with respect to y:

$$f_y = \frac{2}{3(y-t)^{1/3}}$$

For any initial data, (t_0, y_0), the function f is continuous, so by Picard's theorem we are guaranteed that a solution to any initial value problem exists. However, if we consider initial data such that $y_0 = t_0$, then the function f_y is undefined at these points (and thus not continuous), meaning that we cannot apply Picard's theorem for these initial data. On the other hand, f_y is perfectly continuous for all other initial data, so we can guarantee a unique solution to the initial value problem for values of y_0 and t_0 such that $\boxed{y_0 \neq t_0}$.

(c) We can rewrite the differential equation as

$$y' = y(1 - \cos t),$$

which can be solved via separation of variables. This procedure amounts to computing

$$\int \frac{dy}{y} = \int (1 - \cos t) dt + C$$
$$\implies \ln y = t - \sin t + C,$$
$$\implies y = Ce^{t - \sin t}$$

If $y(0) = \sqrt{2}$ then $C = \sqrt{2}$ so that in the long term $(t \to \infty)$, y grows arbitrarily large $y \to \infty$.

3. (20 pts) In this problem we will solve the differential equation

$$\cos(x)\frac{dy}{dx} + \sin(x)y = 1$$

(a) Rewrite the differential equation in the form $y' + p(x)y = f(x)$. Clearly identify $p(x)$ and $f(x)$.

(b) Find the homogeneous solution y_h to the differential equation you found in part (a).

(c) Using the Euler-Lagrange (variation of parameters) method, find the particular solution to the equation you found in part (a).

(d) Solve the initial value problem consisting of the differential equation and the initial condition $y(0) = 1$.

Solution:

(a) Rewriting the equation, we have

$$y' + \tan(x)y = \frac{1}{\cos x},$$
$$p(x) = \tan(x), \quad f(x) = \frac{1}{\cos x}$$

(b) The homogeneous problem is

$$y' + \tan(x)y = 0,$$

which can be solved via separation of variables, so we compute

$$\int \frac{dy}{y} = -\int \frac{\sin x}{\cos x} dx$$

The RHS is solved by a u-substitution $u = \cos x$, $du = -\sin x dx$ so that we have

$$\begin{aligned} \ln y &= \ln(\cos x) + C \\ \implies y &= C \cos x \end{aligned}$$

(c) For the variation of parameters procedure, we guess $y_p = v(x) y_h = v(x)\cos(x)$ and insert this into the nonhomogeneous equation, where we compute

$$\begin{aligned} (v(x)\cos x)' + v(x)\sin(x) &= \frac{1}{\cos x}, \\ \implies v' \cos x - v \sin x + v \sin x &= \frac{1}{\cos(x)}, \\ \implies v' &= \frac{1}{\cos^2 x} = \sec^2 x \\ \implies v &= \tan x \end{aligned}$$

so we have $y_p = v(x)\cos x = \sin x$

(d) If $y(0) = 1$ then we have

$$\begin{aligned} 1 &= C\cos(0) + \sin(0) \\ \implies C &= 1 \end{aligned}$$

so the solution to the IVP is $y(x) = \cos x + \sin x$

4. (24 pts) A tank with a capacity of 50 gal originally contains 10 gal of fresh water. Water containing 1 lb of salt per gallon is entering at a rate of 2 gal/min, and the well mixed mixture is allowed to flow out of the tank at a rate of 1 gal/min. Let $x(t)$ denote the amount of salt in the tank for those values of t such that $0 \le t \le t_{\text{full}}$, where t_{full} is the time that the tank is full.

   (a) Write down an initial value problem that $x(t)$ satisfies.
   (b) Compute an integrating factor to solve the initial value problem in part (a).
   (c) Use the integrating factor method to solve the initial value problem in part (a).
   (d) What is the concentration of salt in the tank for $0 \le t \le t_{\text{full}}$?
   (e) Compare the concentration of salt in the tank at the time the tank begins overflowing $(t = t_{\text{full}})$ to the concentration in the tank if the tank had infinite capacity.

Solution:

(a) The differential equation for the amount of salt is

$$\begin{aligned} \frac{dx}{dt} &= \text{concentration in} \cdot \text{flow rate in} - \text{concentration out} \cdot \text{flow rate out} \\ &= \left(1 \frac{\text{lb}}{\text{gal}}\right)\left(2 \frac{\text{gal}}{\text{min}}\right) - \frac{x(t)\text{lb}}{(10+t)\text{gal}}\left(1 \frac{\text{gal}}{\text{min}}\right) \\ &= 2 - \frac{x}{10+t} \end{aligned}$$

with initial condition $x(0) = 0$.

(b) The integrating factor is

$$\begin{aligned}\mu &= e^{\int \frac{1}{10+t}dt} \\ &= e^{\ln(10+t)} \\ &= 10+t\end{aligned}$$

(c) Using the integrating factor method, we have

$$\begin{aligned}(10+t)x' + x &= 2(10+t) \\ \Longrightarrow \frac{d}{dt}\left[(10+t)x\right] &= 2(10+t)\end{aligned}$$

Integrating with respect to t gives

$$\begin{aligned}(10+t)x &= (10+t)^2 + C \\ \Longrightarrow x(t) &= 10+t+\frac{C}{10+t},\end{aligned}$$

applying the initial condition, $x(0) = 0$ gives

$$\begin{aligned}0 &= 10+0+\frac{C}{10} \\ \Longrightarrow C &= -100\end{aligned}$$

so the amount of salt is given by

$$x(t) = 10+t-\frac{100}{10+t}$$

(d) The concentration in the tank is given by $c(t) = x(t)/V(t)$ where $V(t) = 10+t$ represents the volume of solution in the tank, which is

$$c(t) = 1-\frac{100}{(10+t)^2}$$

(e) The tank overflows at $t = 40$, so the concentration of salt is

$$\begin{aligned}c(t) = 1-\frac{100}{(10+40)^2} &= 1-\frac{100}{2500} \\ &= 1-0.04 \\ &= 0.96,\end{aligned}$$

If the tank were infinitely large, then the concentration of salt in the tank would be 1 lb/gal, so for the finite tank we are close to this theoretical threshold.

5. (20 pts) In this problem, we will study the initial value problem

$$\frac{dy}{dt}+\frac{2}{t}y = t, \quad y(1) = 2.$$

(a) Find the general solution to the differential equation.

(b) Apply the initial condition $y(1) = 2$ to determine the solution of the initial value problem.

(c) Approximate $y(2)$ using one step of Euler's method with $h = 1$.

Solution:

(a) For this equation, $p(t) = 2/t$ so the integrating factor is given by

$$\mu(t) = e^{\int p(t)dt} = t^2.$$

Multiplying the differential equation by $\mu(t)$, we find

$$t^2 y' + 2ty = t^3,$$

which can be rewritten as

$$\frac{d}{dt}\left[t^2 y\right] = t^3.$$

Integrating both sides with respect to t gives

$$\begin{aligned} t^2 y &= \frac{t^4}{4} + C, \\ \implies y &= \frac{1}{4}t^2 + Ct^{-2} \end{aligned}$$

Variation of parameters can also be used to find the general solution.

(b) Applying $y(1) = 2$ gives

$$2 = \frac{1}{4} + C, \implies C = \frac{7}{4}$$

so that the solution of the IVP is

$$y(t) = \frac{1}{4}t^2 + \frac{7}{4}t^{-2}$$

(c) To apply Euler's method, we rewrite the equation as

$$y' = f(t, y),$$

where $f(t, y) = t - \frac{2}{t}y$. The starting points of the iteration are given by

$$t_0 = 1, \quad y_0 = 2,$$

so that with a step size of $h = 1$, we have $t_1 = 2$

$$\begin{aligned} y_1 &= y_0 + 1f(t_0, y_0) \\ &= 2 + 1\left(1 - \frac{2}{1}2\right) \\ &= -1 \end{aligned}$$