APPM 2360 **Exam 1** **Summer 2019**

On the front of your Bluebook write: (1) your name, (2) your section number: **300** for Thaler or **301** for Sprenger and (3) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one sided crib sheet is allowed, which you can take with you when you finish the exam. Make sure to read all instructions carefully and box your final answer.

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1. (16 pts) **True/False** (answer True if it is always true otherwise answer False). No justification is required as there is no partial credit on this question.
   (a) The operator $L[y] = y'' + e^t y' + \cos(t)y$ satisfies the two properties of linear operators.
   (b) A population of bacteria doubles in population every 10 days. The population size is described by the differential equation $\frac{dy}{dt} = 10y$ where $t = 1$ corresponds to 1 day.
   (c) Let $y' = f(y)$, where f is continuous. An $f(y)$ exists such that there are only two equilibrium solutions, both of which are stable.
   (d) The equation $y'' + y^2 = 0$ is a linear, second order, homogeneous differential equation.

2. (20 pts) The following questions are unrelated. Answer each question and justify your response for full credit.
   (a) Compute the equilibrium solutions of the differential equation $y' = (1 - y^2)y$ and classify their stability.
   (b) Given the initial value problem
   $$y' = (y - t)^{2/3}, \quad y(t_0) = y_0,$$
   for which initial conditions (t_0, y_0) are we guaranteed that there exists a unique solution to the initial value problem?
   (c) Solve the differential equation
   $$y' = y - y\cos(t)$$
   What is the long term behavior of the solution if $y(0) = \sqrt{2}$?

3. (20 pts) In this problem we will solve the differential equation
   $$\cos(x)\frac{dy}{dx} + \sin(x)y = 1$$
   (a) Rewrite the differential equation in the form $y' + p(x)y = f(x)$. Clearly identify $p(x)$ and $f(x)$.
   (b) Find the homogeneous solution y_h to the differential equation you found in part (a).
   (c) Using the Euler-Lagrange (variation of parameters) method, find the particular solution to the equation you found in part (a).
   (d) Solve the initial value problem consisting of the differential equation and the initial condition $y(0) = 1$.

There are more problems on the back!

4. (24 pts) A tank with a capacity of 50 gal originally contains 10 gal of fresh water. Water containing 1 lb of salt per gallon is entering at a rate of 2 gal/min, and the well mixed mixture is allowed to flow out of the tank at a rate of 1 gal/min. Let $x(t)$ denote the amount of salt in the tank for those values of t such that $0 \leq t \leq t_{\text{full}}$, where t_{full} is the time that the tank is full.

   (a) Write down an initial value problem that $x(t)$ satisfies.

   (b) Compute an integrating factor to solve the initial value problem in part (a).

   (c) Use the integrating factor method to solve the initial value problem in part (a).

   (d) What is the concentration of salt in the tank for $0 \leq t \leq t_{\text{full}}$?

   (e) Compare the concentration of salt in the tank at the time the tank begins overflowing $(t = t_{\text{full}})$ to the concentration in the tank if the tank had infinite capacity.

5. (20 pts) In this problem, we will study the initial value problem

$$\frac{dy}{dt} + \frac{2}{t}y = t, \quad y(1) = 2.$$

   (a) Find the general solution to the differential equation.

   (b) Apply the initial condition $y(1) = 2$ to determine the solution of the initial value problem.

   (c) Approximate $y(2)$ using one step of Euler's method with $h = 1$.