APPM 2360: Final Exam
July 27, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Textbooks, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **1 side only**) note sheet is allowed.

Problem 1: (40 points) Answer each part of this question with TRUE or FALSE. **DO NOT** write T or F. Also, **provide a brief ONE SENTENCE justification.**

(a) Suppose A is 2×2 and has one repeated real eigenvalue λ. Then λ has two linearly independent eigenvectors.

(b) Picard's Theorem says that there exists a unique solution to $\frac{dy}{dt} = \frac{y}{t}$, $y(1) = 0$ within some open interval containing $t_0 = 1$.

(c) Suppose λ_1 and λ_2 are distinct eigenvalues of a matrix A and $\lambda_1 \neq \lambda_2$. Further assume that $\vec{v}$ and $\vec{u}$ are eigenvectors corresponding to λ_1 and $\vec{w}$ is an eigenvector corresponding to λ_2. Then the set $\{\vec{v}, \vec{u}, \vec{w}\}$ is linearly independent by the Distinct Eigenvalue Theorem.

(d) Suppose 2×2 matrix A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (i.e., $\vec{v}_i$ is an eigenvector corresponding to λ_i for $i = 1, 2$). Then $A \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

(e) Let $V = C([0, 1])$ be the space of continuous functions on $[0, 1]$. Then

$$W = \{f \in C([0, 1]) \mid \int_0^1 f(x)dx = 1\}$$

is a subspace of V.

Solution:

(a) FALSE, may have 1

(b) TRUE, both $\frac{dy}{dt} = f(t, y) = \frac{y}{t}$ and $f_y(t, y) = \frac{1}{t}$ are continuous in a region containing $(t_0 = 1, y_0 = 2)$

(c) FALSE, $\vec{v}, \vec{u}$ may not be linearly independent

(d) TRUE, because

$$A \begin{pmatrix} -1 \\ 2 \end{pmatrix} = A((-1)\vec{v}_1 + 2\vec{v}_2) = (-1)(3)\vec{v}_1 + (-1)(2)\vec{v}_2 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

(e) FALSE, isn't closed under addition or scalar multiplication

Problem 2: (40 points) Solve the following IVP **using Laplace Transforms**.

$$y' - y = f(t), \quad y(0) = 0$$

where

$$f(t) = \begin{cases} 1 & 0 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$f(t) = \text{step}(t) - \text{step}(t - 2)$$

Take Laplace Transform

$$sY - y(0) - Y = \frac{1}{s} - \frac{e^{-2s}}{s}$$
$$\implies Y = \frac{1}{s(s-1)} - \frac{e^{-2s}}{s(s-1)}$$

Partial fractions

$$\frac{A}{s} + \frac{B}{s-1} = \frac{As - A + Bs}{s(s-1)} \implies A = -1, B + A = 0 \implies A = -1, B = 1$$

Thus,

$$\begin{aligned}
\frac{1}{s(s-1)} - \frac{e^{-2s}}{s(s-1)} &= -\frac{1}{s} + \frac{1}{s-1} + e^{-2s}\left(-\frac{1}{s} + \frac{1}{s-1}\right) \\
&= -L(1) + L(e^t) - e^{-2s}L(1) + e^{-2s}L(e^t) \\
&= -L(1) + L(e^t) - e^{-2s}L(1) - L(1 \cdot \text{step}(t-2)) + L(e^{t-2}\text{step}(t-2)) \\
\implies y(t) &= -1 + e^t - \text{step}(t-2) + e^{t-2}\text{step}(t-2)
\end{aligned}$$

Problem 3: (40 points)
Consider the differential equation

$$\ddot{x} + 3\dot{x} - 10x = f(t) \tag{1}$$

(a) Determine the long term behavior of the differential equation when $f(t) = 0$.
(b) Solve the initial value problem for the differential equation when $f(t) = 21e^{2t}$, $x(0) = 3$, and $\dot{x}(0) = 2$.

Solution:

(a) When $f(t) = 0$, we have
$\ddot{x} + 3\dot{x} - 10x = 0$
This differential equation has the characteristic polynomial
$\lambda^2 + 3\lambda - 10 = 0$
with solutions $\lambda = -5, 2$
Given these eigenvalues, the solution to the differential equation will experience exponential growth as $t \to \infty$.

(b) When $f(t) = 3e^{2t}$, we have
$\ddot{x} + 3\dot{x} - 10x = 3e^{2t}$
From above, we know that
$x_h = c_1e^{-5t} + c_2e^{2t}$
We can solve for the particular solution either via variation of parameters or undetermined coefficients.

The solution here will use undetermined coefficients. The particular solution is of the form
$x_p = ate^{2t}$
Substituting this form into the differential equation, we have
$4ae^{2t} + 4ate^{2t} + 3ae^{2t} + 6ate^{2t} - 10ate^{2t} = 21e^{2t}$
Thus,
$7ae^{2t} = 21e^{2t}$ or $a = 3$
Therefore,
$x = x_h + x_p = c_1e^{-5t} + c_2e^{2t} + 3te^{2t}$
Solving the initial value problem, we have

$x(0) = c_1 + c_2 = 3$
$\dot{x}(0) = -5c_1 + 2c_2 + 3 = 2$
so $c_1 = 1$ and $c_2 = 2$.
The solution is thus
$x = e^{-5t} + 2e^{2t} + 3te^{2t}$

Problem 4: (40 points) The following two parts are unrelated.

(a) Consider a series of tanks of brine (salt water) with the following properties:
- Tank A:
  - Size: 50 gallons
  - Outflow:
    10 gallons a minute to Tank B
    10 gallons a minute to Tank C
- Tank B:
  - Size: 30 gallons
  - Outflow:
    15 gallons a minute to Tank A
- Tank C:
  - Size: 35 gallons
  - Outflow:
    5 gallons a minute to Tank A
    5 gallons a minute to Tank B

Set up a system of differential equations to represent the change in salt levels in each tank. Do **NOT** solve this system.

(b) Provide the general solution for the following system of differential equations:

$$\vec{x}' = \begin{pmatrix} 2 & 8 \\ 2 & 2 \end{pmatrix} \vec{x}$$

Solution:

(a) The system can be represented by
$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}' = \begin{bmatrix} -\frac{2}{5} & \frac{1}{2} & \frac{1}{7} \\ \frac{1}{5} & -\frac{1}{2} & \frac{1}{7} \\ \frac{1}{5} & 0 & -\frac{2}{7} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

(b) First we must find the eigenvalues of this system. The characteristic polynomial is
$\det(A - \lambda I) = \lambda^2 - 4\lambda - 12 = 0$
This has solutions $\lambda_1 = -2, \lambda_2 = 6$
Now we need the eigenvectors of this system. Substituting and row reducing, we see that
$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
The solution to this system is therefore given by
$\vec{x} = c_1 e^{-2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Problem 5: (40 points)
Consider the set of vectors

$$\left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix} \right\}$$

(a) Find a basis for $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(b) Let A be the matrix with the vectors above as its columns, i.e., the i^{th} column of A is $\vec{v}_i$. What is the dimension of the column space of A?

(c) Find a basis for and the dimension of the null space of the matrix A from part (b).

Solution:

(a)

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 3 \\ 4 & 1 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(b) By work in part (a), dimension is 2

(c) By work in part (a), $x_1 = -3x_3$, $x_2 = 3x_3$

$$\implies \vec{x} = x_3 \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$$

Table of Laplace Transforms. $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$ where f is of exponential order α

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$, $s > 0$	$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$, $s > 0$	$\mathcal{L}\{\delta(t)\} = 1$, $s > 0$
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$, $s > a$	$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$, $s > 0$	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, $s > \alpha$
$\mathcal{L}\{\text{step}(t)\} = \frac{1}{s}$, $s > 0$	$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s > 0$	$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$, $s > \alpha$
$\mathcal{L}\{f(t-a)\text{step}(t-a)\} = e^{-as}F(s)$, $s > a$	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$	