APPM 2360: Final Exam
July 27, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Textbooks, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **1 side only**) note sheet is allowed.

Problem 1: (40 points) Answer each part of this question with TRUE or FALSE. **DO NOT** write T or F. Also, **provide a brief ONE SENTENCE justification.**

(a) Suppose A is 2×2 and has one repeated real eigenvalue λ. Then λ has two linearly independent eigenvectors.

(b) Picard's Theorem says that there exists a unique solution to $\frac{dy}{dt} = \frac{y}{t}$, $y(1) = 0$ within some open interval containing $t_0 = 1$.

(c) Suppose λ_1 and λ_2 are distinct eigenvalues of a matrix A and $\lambda_1 \neq \lambda_2$. Further assume that $\vec{v}$ and $\vec{u}$ are eigenvectors corresponding to λ_1 and $\vec{w}$ is an eigenvector corresponding to λ_2. Then the set $\{\vec{v}, \vec{u}, \vec{w}\}$ is linearly independent by the Distinct Eigenvalue Theorem.

(d) Suppose 2×2 matrix A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (i.e., $\vec{v}_i$ is an eigenvector corresponding to λ_i for $i = 1, 2$). Then $A\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

(e) Let $V = C([0, 1])$ be the space of continuous functions on $[0, 1]$. Then

$$W = \{f \in C([0, 1]) \mid \int_0^1 f(x)dx = 1\}$$

is a subspace of V.

Problem 2: (40 points) Solve the following IVP **using Laplace Transforms**.

$$y' - y = f(t), \quad y(0) = 0$$

where

$$f(t) = \begin{cases} 1 & 0 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3: (40 points)
Consider the differential equation

$$\ddot{x} + 3\dot{x} - 10x = f(t) \tag{1}$$

(a) Determine the long term behavior of the differential equation when $f(t) = 0$.

(b) Solve the initial value problem for the differential equation when $f(t) = 21e^{2t}$, $x(0) = 3$, and $\dot{x}(0) = 2$.

Problem 4: (40 points) The following two parts are unrelated.

(a) Consider a series of tanks of brine (salt water) with the following properties:
- Tank A:
  - Size: 50 gallons
  - Outflow:
    10 gallons a minute to Tank B
    10 gallons a minute to Tank C

- Tank B:
  - Size: 30 gallons
  - Outflow:
    15 gallons a minute to Tank A
- Tank C:
  - Size: 35 gallons
  - Outflow:
    5 gallons a minute to Tank A
    5 gallons a minute to Tank B

Set up a system of differential equations to represent the change in salt levels in each tank. Do **NOT** solve this system.

(b) Provide the general solution for the following system of differential equations:

$$\vec{x}' = \begin{pmatrix} 2 & 8 \\ 2 & 2 \end{pmatrix} \vec{x}$$

Problem 5: (40 points)
Consider the set of vectors

$$\left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix} \right\}$$

(a) Find a basis for span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
(b) Let A be the matrix with the vectors above as its columns, i.e., the i^{th} column of A is $\vec{v}_i$. What is the dimension of the column space of A?
(c) Find a basis for and the dimension of the null space of the matrix A from part (b).

Table of Laplace Transforms. $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$ where f is of exponential order α

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$, $s > 0$	$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$, $s > 0$	$\mathcal{L}\{\delta(t)\} = 1$, $s > 0$
$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, $s > a$	$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$, $s > 0$	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, $s > \alpha$
$\mathcal{L}\{\text{step}(t)\} = \frac{1}{s}$, $s > 0$	$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s > 0$	$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$, $s > \alpha$
$\mathcal{L}\{f(t-a)\text{step}(t-a)\} = e^{-as}F(s)$, $s > a$	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$	