APPM 2360: Midterm Exam 3

July 13, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Textbooks, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **1 side only**) note sheet is allowed.

Problem 1: (40 points) Answer each part of this question with TRUE or FALSE. **DO NOT** write T or F. Also, **provide a brief ONE SENTENCE justification.**

(a) The harmonic oscillator modeled by the differential equation

$$2\ddot{x} + 5\dot{x} + 3x = e^{-3t}$$

will not experience resonance.

(b) A basis for the solution space for the differential equation

$$y'' + 2y' + y = 0$$

has three linearly independent functions.

(c) For the forced oscillator $m\ddot{x} + b\dot{x} + kx = F\cos(\omega_f t)$, beats occur when the forcing frequency, ω_f, is equal to the natural frequency, $\omega_0 = \sqrt{k/m}$.

(d) The method of undetermined coefficients can be used to find a particular solution to $t^2y'' + 3y = \cos t$

Solution:

(a) TRUE - The forcing function is not periodic/a sine or cosine

(b) FALSE - Only two function will be in the basis

(c) FALSE - Beats occur when the are NOT equal

(d) FALSE - It doesn't have constant coefficients

Problem 2: (40 points) The following problems are unrelated.

(a) Is $\{e^{-2t}, e^{t}\}$ a basis for the solution space of $\ddot{x} + 5\dot{x} + 6x = 0$? Be sure to show your work and justify your answer. Don't just say "Yes" or "No".

(b) For a mass-spring system $m\ddot{x} + b\dot{x} + kx = 0$, state the three types of damped motion and, in each case, provide the equation in terms of m, b, and k for when that type of motion occurs. In each of the three cases, your answer should be in the form "_____-damped motion occurs when _____" where you fill in the blank spaces.

(c) Find the general solution for the following homogeneous linear differential equation

$$y^{(4)} + 4y''' + 3y'' = 0$$

Solution:

(a) The characteristic equation is $r^2 + 5r + 6 = (r+3)(r+2) = 0$ so a basis for the solution space is $\{e^{-2t}, e^{-3t}\}$. So the given set is NOT a basis.

(b) Over-damped motion occurs when $b^2 - 4mk > 0$. Critically-damped motion occurs when $b^2 - 4mk = 0$. Under-damped motion occurs when $b^2 - 4mk < 0$

(c) The characterisitc equation is $r^4 + 4r^3 + 3r^2 = r^2(r^2 + 4r + 3) = r^2(r+3)(r+1)$. Thus, the general solution is

$$y(t) = c_1 + c_2 t + c_3 e^{-3t} + c_4 e^{-t}$$

Problem 3: (40 points)
Consider the differential equation

$$x'' - 4x' + 9x = f(t)$$

(a) Find the homogeneous solution for this differential equation.
(b) Given the following forcing functions, provide the initial guess for the form of the particular solutions for undetermined coefficients. **DO NOT** solve for the particular solution.
  (i) $f(t) = t^2 + 3$
  (ii) $f(t) = e^{2t}\sin\left(\sqrt{5}t\right)$
  (iii) $f(t) = t^2 e^t$
(c) Use **variation of parameters** to solve for the particular solution when $f(t) = e^{2t}$.

Solution:

(a) The characteristic polynomial for this ODE is
  $\lambda^2 - 4\lambda + 9 = 0$
  This equation has roots $\lambda = 2 \pm \sqrt{5}i$ so the homogeneous solution is given by
  $x_h = c_1 e^{2t}\cos\left(\sqrt{5}t\right) + c_2 e^{2t}\sin\left(\sqrt{5}t\right)$
(b) (i) The forcing fuction is a polynomial of degree two, so
    $x_p = at^2 + bt + c$
  (ii) The forcing function is in the homogeneous solution space, so
    $x_h = ate^{2t}\cos\left(\sqrt{5}t\right) + bte^{2t}\sin\left(\sqrt{5}t\right)$
  (iii) The forcing function is a product of an exponential and a polynomial, so
    $x_h = at^2e^t + bte^t + ce^t$
(c) The particular solution will be of the form $x_p = v_1x_1 + v_2x_2$, where $v_i = \int \frac{w_i}{w}dt$.

  Using the solution from (a), we have
  $w = \begin{vmatrix} e^{2t}\cos\left(\sqrt{5}t\right) & e^{2t}\sin\left(\sqrt{5}t\right) \\ 2e^{2t}\cos\left(\sqrt{5}t\right) - \sqrt{5}e^{2t}\sin\left(\sqrt{5}t\right) & 2e^{2t}\sin\left(\sqrt{5}t\right) \end{vmatrix} + \sqrt{5}e^{2t}\cos\left(\sqrt{5}t\right)$
  Thus, $w = \sqrt{5}e^{4t}$.
  In the second order case, $w_1 = -fx_2$ and $w_2 = fx_1$. Therefore,
  $v_1 = -\int \frac{e^{4t}\sin\left(\sqrt{5}t\right)}{\sqrt{5}e^{4t}}dt = \frac{1}{5}\cos\left(\sqrt{5}t\right)$
  and
  $v_2 = \int \frac{e^{4t}\cos\left(\sqrt{5}t\right)}{\sqrt{5}e^{4t}}dt = \frac{1}{5}\sin\left(\sqrt{5}t\right)$
  Substituting, we have $x_p = v_1x_1 + v_1x_2 = \frac{1}{5}e^{2t}\cos^2\left(\sqrt{5}t\right) + \frac{1}{5}e^{2t}\sin^2\left(\sqrt{5}t\right)$ or $x_p = \frac{1}{5}e^{2t}$.

Problem 4: (40 points) There are "runner's highs" and "runner's lows" when you feel elated and terrible, respectively, on a long run. Suppose your mood on a long run can be modeled as an oscillator:

$$\ddot{x} + 4\dot{x} + 5x = F\cos(\omega_f t)$$

(a) What is the general solution to the associated homogeneous problem: $\ddot{x} + 4\dot{x} + 5x = 0$?
(b) Consider the case when your willpower provides a constant amount of "force", $F > 0$, to keep you feeling good. In this case, your mood is modeled by

$$\ddot{x} + 4\dot{x} + 5x = F\,, \qquad F > 0$$

  (i) Find the general solution to this nonhomogeneous equation.
  (ii) What is the long time behavior of the solution? That is, what is $\lim_{t\to\infty} y(t)$?
(c) Now suppose your willpower provides a varying amount of "force", $F(t) = \cos t$. Find the solution to the associated IVP given by

$$\ddot{x} + 4\dot{x} + 5x = \cos t\,, \qquad x(0) = 1\,, \qquad \dot{x}(0) = 0$$

Solution:

(a) Characteristic equation is $r^2 + 4r + 5 = 0$ which has roots

$$r = -\frac{4}{2} \pm \frac{\sqrt{4^2 - 4(5)}}{2} = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm i$$

So the general homogeneous solution is

$$x_h = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$$

(b) (i) Using method of undetermined coefficients particular solutions guess is $x_p = A$. Plugging in gives

$$5A = F \implies A = \frac{F}{5}$$

So general solution is

$$x = x_h + x_p = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t + \frac{F}{5}$$

(ii)

$$\lim_{t\to\infty} x(t) = \frac{F}{5}$$

(c) Using method of undetermined coefficients particular solutions guess is $x_p = A\cos t + B\sin t$. Plugging in gives

$$\begin{aligned}\cos t = \ddot{x}_p + 4\dot{x}_p + 5x_p &= -A\cos t - B\sin t + 4(-A\sin t + B\cos t) + 5(A\cos t + B\sin t)\\ &= \cos t(-A + 4B + 5A) + \sin t(-B - 4A + 5B)\\ &= \cos t(4A + 4B) + \sin t(4B - 4A)\end{aligned}$$

$$\implies 4B - 4A = 0 \implies A = B \quad \text{and} \quad 8A = 1 \implies A = B = \frac{1}{8}$$

So general solution is

$$x = x_h + x_p = c_1 e^{-2t} \cos\left(\frac{3}{2}t\right) + c_2 e^{-2t} \sin\left(\frac{3}{2}t\right) + \frac{1}{8}(\cos t + \sin t)$$

Using the ICs gives

$$x(0) = 1 = c_1 + \frac{1}{8} \implies c_1 = \frac{7}{8}$$

$$\dot{x} = e^{2t}(-2c_1\cos t + c_2\cos t - c_1\sin t - 2c_2\sin t)$$

$$\dot{x}(0) = 0 = -2c_1 + c_2 + \frac{1}{8} = -2\frac{7}{8} + c_2 + \frac{1}{8} \implies c_2 = \frac{13}{8}$$

Thus, solution to IVP is

$$x = \frac{7}{8}e^{-2t}\cos t + \frac{13}{8}e^{-2t}\sin t + \frac{1}{8}(\cos t + \sin t)$$

Problem 5: (40 points)
Consider the differential equation

$$\ddot{x} + b\dot{x} + \frac{9}{4}x = 0 \tag{1}$$

(a) Find the roots of the characteristic polynomial for this differential equation.

(b) Provide a general solution for the differential equation. Note: There are two values of the parameter b for which the differential equation is different than all other values of b. Be sure to address this case separately.

(c) Describe all different possible behaviors of the solution to this differential equation based upon the value of b. That is to say, give values of b such that

(i) the solution is or is not oscillatory

(ii) the solution grows or decays

Solution:

(a) The characteristic polynomial is $\lambda^2+b\lambda+\frac{9}{4}=0$. This polynomial has roots $\lambda=-\frac{b}{2}\pm\frac{\sqrt{b^2-9}}{2}$.

(b) There are two (or three) cases to consider:

(i) $b=\pm 3$

When $b=\pm 3$, we have repeated real roots. Thus,

$x=c_1e^{-\frac{b}{2}t}+c_2te^{-\frac{b}{2}t}$

(ii) The solution can otherwise be given by

$x=c_1e^{\left(-\frac{b}{2}+\frac{\sqrt{b^2-9}}{2}\right)t}+c_2te^{\left(-\frac{b}{2}-\frac{\sqrt{b^2-9}}{2}\right)t}$

(iii) When $-3<b<3$, λ is complex and the solution may alternatively be given by

$x=c_1e^{-bt/2}\cos\left(\frac{\sqrt{9-b^2}}{2}t\right)+c_2te^{-bt/2}\sin\left(\frac{\sqrt{9-b^2}}{2}t\right)$

(c) (i) The solution will oscillate when $-3<b<3$. The solution will not oscillate when $b\leq -3$ or $b\geq 3$.

(ii) The solution will grow when $b<0$. The solution will decay when $b>0$. The solution will neither grow nor decay when $b=0$.