APPM 2360: Midterm Exam 2
June 29, 2018

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Textbooks, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **1 side only**) note sheet is allowed.

Problem 1: (40 points) Answer each part of this question with TRUE or FALSE. **DO NOT** write T or F. Also, **provide a brief ONE SENTENCE justification.**

(a) The set of continuous functions on $[0, 1]$ satisfying $f(0) = 0$ is a subspace of the set of continuous functions on $[0, 1]$.
(b) For any $n \times n$ matrices A and B, $|AB| = |BA|$.
(c) Suppose A and B are both invertible $n \times n$ matrices. Then the solution to $BA\vec{x} = \vec{b}$ is $\vec{x} = B^{-1}A^{-1}\vec{b}$.
(d) If A is a 3×4 matrix with 3 pivots, then there exist $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has no solution.
(e) Three linearly independent vectors in $\mathbb{R}^3$ always span $\mathbb{R}^3$.

Solution:

(a) TRUE, satisfies two closure properties
(b) TRUE, $|AB| = |A||B| = |B||A| = |BA|$
(c) FALSE, solution is $A^{-1}B^{-1}\vec{b}$
(d) FALSE, there is a solution for any $\vec{b}$ since A has 3 pivots
(e) TRUE, dimension of $\mathbb{R}^3$ is 3 so any 3 linearly independent vectors span $\mathbb{R}^3$ by definition of dimension

Problem 2: (40 points) The following problems are unrelated.

(a) Determine whether the following matrix is invertible or not. State explicitly whether the matrix is invertible or singular.

$$A = \begin{bmatrix} 0 & 3 & 5 \\ 5 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}$$

(b) Determine whether $W = \{2 \times 2 \text{ matrices with determinant equal to } 0\}$ is a subspace of the vector space $V = \mathbb{M}_{2\times 2} = \{2 \times 2 \text{ matrices}\}$

Solution:

(a)

$$|A| = -5 \cdot \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = -5(9 - 10) + 3(6 - 5) = 5 + 3 = 8$$

So invertible

(b) Not a subspace. $X = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are both 2×2 determinant 0 matrices, i.e., they are in W. But $X + Y$ has determinant 1 so it is not in W. So W is not closed under addition so it is not a subspace

Problem 3: (40 points)

(a) Given the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{bmatrix}$ and the vector $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ find **ONE** solution to the equation $A\vec{x} = \vec{b}$.

(b) Let B be a 3×3 matrix of real numbers. The equation $B\vec{x} = \vec{c}$ is satisfied by $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$, for some vector $\vec{c}$. Suppose B has two pivot columns when in reduced row echelon form. Find **ALL** solutions to $B\vec{x} = \vec{0}$.

Solution:

(a) $RREF\left(A|\vec{b}\right) = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus one solution is given by $\vec{x}_p = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$

Note: Multiple answers are possible given the choice of free variable

(b) All particular solutions differ by a homogeneous solution. Thus, $\vec{x}_h = c\left(\vec{x}_1 - \vec{x}_2\right)$ or

$\vec{x}_h = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

olutions), provide all solutions to the equation $A\vec{x} = \vec{b}$ from part (a).

Problem 4: (40 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 3 & 2 & -1 & 5 \end{bmatrix}$$

(a) Find a basis for the column space of A, the span of the columns of A. That is to say, find a basis for the vector space of all vectors $\vec{b}$ such that the equation $A\vec{x} = \vec{b}$ is consistent.

(b) Find a basis for the null space of A. That is to say, find a basis for the vector space of all vectors $\vec{x}$ such that the equation $A\vec{x} = \vec{0}$ is satisfied.

(c) Briefly verify that any linear combination of the basis vectors given in part (b) satisfies $A\vec{x} = \vec{0}$.

Solution:

$$RREF\,(A) = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The basis is given by the pivot columns. Thus, $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

(b) The basis is given by setting each free variable to 1 one at a time while letting the other equal 0. Thus, $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Note: Due to multiple valid choices of scaling, there may be equivalent answers.

(c) Let $\vec{x}_1$ and $\vec{x}_2$ be the two basis vectors from part (b). Then,

$A(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1 A\vec{x}_1 + c_2 A\vec{x}_2$

$A(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1 \cdot 0 + c_2 \cdot 0$

$A(c_1\vec{x}_1 + c_2\vec{x}_2) = 0$

Problem 5: (40 points) The following problems are unrelated.

(a) For what value(s) of a and b, if any, can you conclude that the following set of functions is linearly independent for $t \in \mathbb{R}$

$$\left\{ e^{at},\ e^{bt},\ 1 \right\}$$

(b) (i) Determine whether the following set of vectors is linearly independent or linearly dependent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right\}$$

(ii) Find a basis for and the dimension of

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right\}$$

Note that the vectors here are the same ones as in part (i).

Solution:

(a)

$$\begin{vmatrix} e^{at} & e^{bt} & 1 \\ ae^{at} & be^{bt} & 0 \\ a^2e^{at} & b^2e^{bt} & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} ae^{at} & be^{bt} \\ a^2e^{at} & b^2e^{bt} \end{vmatrix} = ab^2e^{(a+b)t} - ba^2e^{(a+b)t} = e^{(a+b)t}(ab^2 - ba^2)$$

which is only identically zero if $ab^2 - ba^2 = ab(b-a) = 0$. So linearly independent as long as $b \neq 0, a \neq 0$, and $b \neq a$.

(b) (i)

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 6 & 4 \\ 2 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the set is linearly dependent

(ii) Using result in part (i), the dimension is 2 and a basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} \right\}$$