APPM 2360: Midterm Exam 1
June 15, 2018

On the front of your Bluebook write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Textbooks, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **1 side only**) note sheet is allowed.

Problem 1: (40 points) Answer each part of this question with TRUE or FALSE. **DO NOT** write T or F. Provide a brief ONE SENTENCE justification.

(a) Let $y' - 2ty^2 = 0$. We can write this equation as $L(y) = 0$ where $L(y) := y' - 2ty^2$ is a linear operator.
(b) Picard's Theorem states that a solution exists but is not unique for the initial value problem $y' = y^{2/3}$, $y(0) = 1$.
(c) If $y_1(t)$ and $y_2(t)$ are both solutions to the differential equation $t^2y' + \sin(t)\,y = \frac{t}{\sqrt{1+t^2}}$, then $y_1(t) + 2y_2(t)$ is also a solution.
(d) The differential equation $y' = 2y - t$ has an equilibrium when $y = \frac{1}{2}t$.

Solution:

(a) FALSE, L is not linear because of y^2
(b) FALSE, $f(t, y) = y^{2/3}$ and $f_y(t, y) = \frac{1}{y^{1/3}}$ are both contituous
(c) FALSE, the linear DE is nonhomogeneous so superposition principle doesn't apply
(d) FALSE, equilibrium solutions are $y(t) =$ constant

Problem 2: (40 points)

(a) What are the two conditions a linear operator, L, must satisfy?
(b) Consider the following word bank

autonomous	linear	nonlinear	homogeneous
nonhomogeneous	separable	1st order	2nd order

For each of the following differential equations, write **all** of the words from the word bank that correctly describe the differential equation.

(i) $\frac{dy}{dt} = \frac{y}{t}$
(ii) $y' = t - ty$
(iii) $\frac{d^2y}{dt^2} = \sin(t + y)$
(iv) $\frac{dy}{dt} = y^2 - 2y$

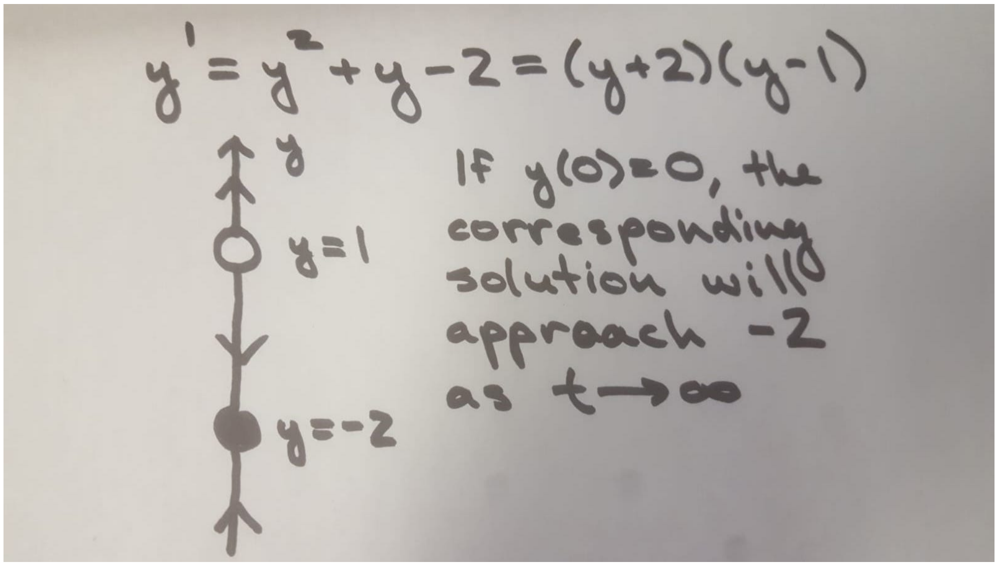
(c) Draw the phase line corresponding to the differential equation

$$\frac{dy}{dt} = y^2 + y - 2$$

Remember, phase lines show any equilibria, each equilibrium's stability, and slope arrows. Just using the phase line (i.e., without computing the solution), determine the long-term behaviour of the solution to the equation with the initial condition $y(0) = 0$.

Solution:

(a) (1) $L(ky) = kL(y)$ and (2) $L(y_1 + y_2) = L(y_1) + L(y_2)$
(b) (i) Linear, 1st order, homogeneous
(ii) Linear 1st order, nonhomogeneous, separable
(iii) Nonlinear, 2nd order
(iv) Nonlinear, 1st order, autonomous, separable

(c) 

Problem 3: (40 points) The following two problems are unrelated.

(a) A holding tank that catches pollutant runoff from some chemical process initially contains 10 liters of water with an initial pollutant concentration of 0.2 grams/liter. Polluted water flows into the tank at a rate of 2 liters/hour and contains 1 gram/liter of pollution in it. A well-mixed solution leaves the tank at 4 liters/hour.

(i) Set up the IVP describing the amount of chemical pollutant (in grams) in the tank after t hours.

(ii) Find the solution to the IVP.

(b) A certain type of bacteria doubles in population every 6.5 hours. Assume the bacteria population grows exponentially.

(i) What is the growth constant k?

(ii) How long until the population triples?

Solution:

(a) (i) $x(0) = 0.2 \cdot 10 = 2.$

$$\frac{dx}{dt} = c_{in}r_{in} - c_{out}r_{out} = 2 \cdot 1 - \frac{x}{10-2t} \cdot 4 = 2 - \frac{4x}{10-2t}$$

$$\implies \frac{dx}{dt} + \frac{4x}{10-2t} = 2$$

(ii) Homogeneous solution:

$$\frac{dx}{4x} = -\frac{dt}{10-2t} \implies \frac{1}{4}\ln|4x| = \frac{1}{2}\ln|10-2t| + C \implies \ln|4x| = 2\ln|10-2t| + C$$

$$\implies x_h(t) = e^{\ln(|10-2t|^2)+C} = C(10-2t)^2$$

Particular solution:

$$\mu(t) = e^{\int \frac{4}{10-2t}dt} = e^{4(-\frac{1}{2}\ln|10-2t|)} = \frac{1}{(10-2t)^2}$$

$$\mu(t) \cdot DE \to \frac{1}{(10-2t)^2}x' + \frac{4x}{(10-2t)^3} = \frac{2}{(10-2t)^2}$$

$$\implies \frac{1}{(10-2t)^2}x = 2\int (10-2t)^{-2}dt = 2\frac{1}{2(10-2t)} + C = \frac{1}{10-2t} + C$$

Choose $C = 0$. So $x_p(t) = 10 - 2t$.

General solution is

$$x = x_h + x_p = C(10-2t)^2 + 10 - 2t$$

$x(0) = 2$ gives

$$x(0) = 2 = C(10)^2 + 10 \implies C = \frac{-8}{100}$$

So

$$x(t) = -\frac{8}{100}(10-2t)^2 + 10 - 2t$$

(b) (i) $y' = ky$. We know $y(t) = y_0 e^{kt}$.

$$y(6.5) = 2y_0 = y_0 e^{6.5 \cdot k} \implies k = \frac{\ln(2)}{6.5}$$

(ii) Population triples when $y(T) = 3y_0$.

$$y(T) = 3y_0 = y_0 e^{\frac{\ln(2)\cdot T}{6.5}}$$

$$\implies T = \frac{\ln(3)\cdot 6.5}{\ln(2)}$$

Problem 4: (40 points)
Consider the differential equation

$$ty' - 2t\sin(t)\cos(t)\,y = -4t\cos(t)\sin(t) \tag{1}$$

(a) Find the general solution to the homogeneous equation associated with Eq. 1.
(b) Find a particular solution for Eq. 1.
(c) Find the general solution for Eq. 1.
(d) Find the solution to Eq. 1 that goes through the point $(\pi, 5)$.

Solution:

(a) $\frac{1}{y}dy = 2\sin(t)\cos(t)\,dt$

$y_h = c_1 e^{(\sin(t))^2}$

(b) $y_p = 2$
(c) $y = y_p + y_h = c_1 e^{(\sin(t))^2} + 2$
(d) $y = 3e^{(\sin(t))^2} + 2$

Problem 5: (40 points)
Consider the differential equation

$$y' = ty^2 + t^2 y \tag{2}$$

(a) Explain why we cannot solve this equation quantitatively given the techniques in the course thus far.
(b) Use three steps of Euler's Method to approximate $y(3)$ given that $y(0) = 1$.
(c) Give two ways to reduce the error in our numerical estimation of the solution.

Solution:

(a) This equation is neither separable nor linear.

(b)

n	t_n	y_n	y'_n	y_{n+1}
0	0	1	0	1
1	1	1	2	3
2	2	3	30	33
3	3	33		

(c) You could

(i) Take more steps of Euler

(ii) Use Improved Euler

(iii) Use RK-4