- This exam is worth 150 points and has 9 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on both sides.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/050724 (16 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given. Please write your answers in a single column separate from any work you do to arrive at the answer.

   (a) The eigenvalues of $\begin{bmatrix} 1 & a & 0 \\ 0 & 2 & b \\ 0 & 0 & 3 \end{bmatrix}$, $a, b \in \mathbb{R}$, are $1, 2, 3$, regardless of the values of a and b.

   (b) If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular square matrices of the same order, then $\left(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{A}$.

   (c) Let $\mathbb{V}$ be a vector space. Any set of vectors in $\mathbb{V}$ that contains the zero vector must be linearly dependent.

   (d) For $n \times n$ matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$, $\operatorname{Tr}\left(\mathbf{A} + \mathbf{B}^{\mathrm{T}} + \mathbf{C}\right) = \operatorname{Tr}\mathbf{A}^{\mathrm{T}} + \operatorname{Tr}\mathbf{B} + \operatorname{Tr}\mathbf{C}^{\mathrm{T}}$.

   (e) If $\mathbf{A}$ is an $m \times n$ matrix where $m \neq n$, then $\left|\mathbf{A}^{\mathrm{T}}\mathbf{A}\right|$ is not defined.

   (f) The equation $e^{w+z}\csc z \dfrac{\mathrm{d}w}{\mathrm{d}z} - wz\sin w = 0$ is separable.

   (g) The equation $y' = y^4 + 4y^2$ has an unstable equilibrium solution.

   (h) There are no values of t where the trajectories of the system $\begin{aligned} x' &= x^2 + y^4 + 4 \\ y' &= e^{x-y} - 1 \end{aligned}$ have horizontal tangents.

2. [2360/050724 (14 pts)] Let $\vec{\mathbf{p}} = 4t^2 + 8t + 5$ and let $\vec{\mathbf{p}}_1 = t^2 + t + 1$, $\vec{\mathbf{p}}_2 = 3t^2 + 3t + 4$ and $\vec{\mathbf{p}}_3 = 2t^2 + 3t + 2$. Is $\vec{\mathbf{p}} \in \operatorname{span}\{\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3\}$? If so, write it as a linear combination of $\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3$. If not, explain why not. Justify your answer.

3. [2360/050724 (18 pts)] Consider the function $f(t) = \begin{cases} 0 & t < 0 \\ 2t & 0 \le t < 2 \\ 4 - (t-2)^2 & t \ge 2 \end{cases}$

   (a) (6 pts) Draw a well labeled graph of $f(t)$.

   (b) (6 pts) Write $f(t)$ using step functions.

   (c) (6 pts) Find the Laplace transform of $f(t)$.

4. [2360/050724 (14 pts)] You and your roommate are concocting an elixir that will help you pass your differential equations exam. To make it, you have two 10-gallon tanks. Initially, Tank 1 is filled with fresh water and Tank 2 is half filled with well-mixed solution in which 7 grams of the miracle spice *diffyQ* is dissolved. For $t > 0$, solution containing 2 grams of *diffyQ* per gallon enters Tank 1 at 3 gallons per hour and this well-mixed solution flows from Tank 1 into Tank 2 at the rate of 4 gallons per hour. Well-mixed solution from Tank 2 flows into Tank 1 at 1 gallon per hour and exits Tank 2 at 2 gallons per hour. The elixir will be perfect when Tank 2 is full. Write a system of differential equations using matrices and vectors that models this initial value problem, providing an appropriate t interval over which the solution is valid.

5. [2360/050724 (20 pts) Use Laplace Transforms to solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} + 4y = 10te^{-4t} + \delta(t-2)$, $y(0) = 0$.

6. [2360/050724 (20 pts)] Solve the initial value problem $t\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(1) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $t > 0$. Assume solutions are of the form $t^{\lambda}\vec{\mathbf{v}}$ as was done in the homework and write your final answer as a single vector.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE ON REVERSE

7. [2360/050724 (16 pts)] Consider an harmonic oscillator having a mass attached to a spring. For the following parts, write, but **DO NOT SOLVE**, the appropriate differential equation that governs the displacement, $x(t)$, with the given description.

(a) (4 pts) The oscillator is unforced, experiences a damping force proportional to twice the instantaneous velocity, has circular (angular) frequency of 3 s^{-1} and a mass of 3 kg.

(b) (4 pts) The oscillator is forced by $10\cos 2t$, has a restoring (spring) constant of 4 N/m and is in resonance.

(c) (4 pts) The oscillator is unforced, critically damped, has mass of 2 kg and a restoring (spring) constant of 2 N/m.

(d) (4 pts) The oscillator is undamped, has a spring (restoring) constant of 3 N/m and circular (angular) frequency of 2 s^{-1} and is forced by $\sin t$ for $\pi \le t \le 2\pi$ and unforced otherwise. Write the forcing function as a single function, not as a piecewise defined function.

8. [2360/050724 (22 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -4 & k \\ 3 & 6 & 12 \end{bmatrix}$ and the vector $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$.

(a) (6 pts) There is one real value of k that makes the matrix $\mathbf{A}$ have a single pivot column. Find it, making sure you are correct because the remaining parts depend on it.

(b) (12 pts) Using the value of k you found in part (a):

i. (4 pts) Find a particular solution of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

ii. (4 pts) Find a basis for the solution space of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ and state its dimension.

iii. (4 pts) Is $\vec{\mathbf{b}} \in$ Col $\mathbf{A}$? Explain briefly.

(c) (4 pts) Is $\mathbf{A}$ invertible, regardless of the value of k? Explain briefly.

9. [2360/050724 (10 pts)] Consider the linear system of differential equations given by $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & k \end{bmatrix}$ and $k \in \mathbb{R}$. For each of the following, find all values of k such that the system exhibits the given behavior. If there are no values of k satisfying the requirements, write NONE. Hint: The fact that $x^2 - 4x + 20 = (x-2)^2 + 16$ may be of help.

(a) The equilibrium point at $(0,0)$ is a degenerate node.

(b) The system has nonisolated equilibrium points.

(c) The equilibrium point at $(0,0)$ is a center.

(d) The equilibrium point at $(0,0)$ is stable.

(e) The equilibrium point at $(0,0)$ is an unstable node.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \ge 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$