1. [2360/041724 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) $7\ddot{x} + 6\dot{x} + e^x + \cos x = 0$ describes a conservative system.

   (b) If y_1, y_2, y_3 are solutions to a third order linear, homogeneous differential equation, then the solution space must be equal to span $\{y_1, y_2, y_3\}$.

   (c) $\mathscr{L}\{t\cosh t\} = \dfrac{s^2+1}{(s^2-1)^2}$

   (d) If the characteristic equation of the differential equation governing a forced oscillator is $2r^2 + 2r + 1 = 0$, and the oscillator is forced by $f(t) = 10\cos\left(\frac{t}{2}\right)$, then the oscillator is in resonance.

   (e) If the charge in a circuit is given by $q(t) = e^{-4t}(\cos t + \sin t) + \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$, then the value of the steady state charge when $t = \frac{\pi}{4}$ is $\frac{1}{2}$.

   SOLUTION:

   (a) **FALSE** Due to the presence of the $\dot{x}$ term, the system is damped and therefore not conservative.

   (b) **FALSE** This will only be the case if $W[y_1, y_2, y_3](t)$ is not identically zero, that is, the functions must be linearly independent.

   (c) **TRUE**

$$\mathscr{L}\{t\cosh t\} = (-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2-1}\right) = -1\left[\frac{(s^2-1)(1) - s(2s)}{(s^2-1)^2}\right] = \frac{s^2+1}{(s^2-1)^2}$$

   (d) **FALSE** Based on the characteristic equation, specifically the $2r$ term, the oscillator is damped and therefore cannot be in resonance.

   (e) **TRUE** The steady state part of the solution is $q_{ss}(t) = \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$ which gives $q_{ss}\left(\frac{\pi}{4}\right) = \frac{1}{2}\sin 2\left(\frac{\pi}{4}\right) - \frac{1}{2}\cos 2\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

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2. [2360/041724 (24 pts)] Use variation of parameters to find the general solution of $t^2u'' + tu' - 9u = 72(t+1)$, assuming solutions of the associated homogeneous equation are of the form $u = t^r$. Be sure to simplify your final answer.

   SOLUTION:

$$u = t^r \implies u' = rt^{r-1} \implies u'' = r(r-1)t^{r-2}$$

$$t^2u'' + tu' - 9u = t^2 r(r-1)t^{r-2} + trt^{r-1} - 9t^r = \left(r^2 - 9\right)t^r = 0$$

$$\implies (r+3)(r-3) = 0 \implies r = 3, -3 \implies u_1 = t^3, u_2 = t^{-3} \implies u_h = c_1t^3 + c_2t^{-3}$$

$$W\left(t^3, t^{-3}\right) = \begin{vmatrix} t^3 & t^{-3} \\ 3t^2 & -3t^{-4} \end{vmatrix} = -6t^{-1}$$

The proper form of the nonhomogeneous term for use in variation of parameters is $\dfrac{72(t+1)}{t^2} = 72\left(t^{-1} + t^{-2}\right)$.

$$v_1' = \frac{-u_2 f}{W} = \frac{-t^{-3}(72)\left(t^{-1} + t^{-2}\right)}{-6t^{-1}} = 12\left(t^{-3} + t^{-4}\right) \implies v_1 = \int 12\left(t^{-3} + t^{-4}\right)\mathrm{d}t = -6t^{-2} - 4t^{-3}$$

$$v_2' = \frac{u_1 f}{W} = \frac{t^3(72)\left(t^{-1} + t^{-2}\right)}{-6t^{-1}} = -12\left(t^3 + t^2\right) \implies v_2 = \int -12\left(t^3 + t^2\right)\mathrm{d}t = -3t^4 - 4t^3$$

$$u_p = v_1u_1 + v_2u_2 = \left(-6t^{-2} - 4t^{-3}\right)t^3 + \left(-3t^4 - 4t^3\right)t^{-3}$$

$$= -6t - 4 - 3t - 4 = -9t - 8$$

$$u(t) = u_h + u_p = c_1t^3 + c_2t^{-3} - 9t - 8$$

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3. [2360/041724 (24 pts)] Consider the initial value problem $2\ddot{x}+6\dot{x}+4x=f(t), x(0)=1, \dot{x}(0)=-8$ describing a certain mass/spring harmonic oscillator.

(a) (2 pts) Is the oscillator underdamped, overdamped or critically damped? Justify your answer.

(b) (10 pts) Assuming that oscillator is unforced/undriven, $f(t)=0$, determine if and when the mass will pass through the equilibrium position.

(c) (10 pts) Suppose now that the oscillator is forced by $f(t)=4e^{-t}$. Using the same initial conditions, use the Method of Undetermined Coefficients to find the position of the mass when $t=2$.

(d) (2 pts) Now suppose you have the ability to change the value of the mass, m, in the mass/spring system. Find all the values of m, if any, that will allow the mass to pass through its equilibrium position more than once, assuming the damping and restoring constants are their original values.

SOLUTION:

(a) Overdamped: $b^2-4mk=6^2-4(2)(4)=36-32=4>0$

(b) Solve the initial value problem.

$$2r^2+6r+4=2(r^2+3r+2)=2(r+2)(r+1)=0 \implies r=-2,-1$$

$$x(t)=c_1e^{-2t}+c_2e^{-t} \implies x(0)=c_1+c_2=1$$

$$\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t} \implies \dot{x}(0)=-2c_1-c_2=-8$$

$$c_1=\frac{\begin{vmatrix}1 & 1\\ -8 & -1\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -2 & -1\end{vmatrix}}=7 \qquad c_2=\frac{\begin{vmatrix}1 & 1\\ -2 & -8\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -2 & -1\end{vmatrix}}=-6$$

$$x(t)=7e^{-2t}-6e^{-t}$$

Now see if there exists a t where the solution vanishes.

$$7e^{-2t}-6e^{-t}=0$$

$$e^{-2t}\left(7-6e^{t}\right)=0$$

$$e^t=\frac{7}{6}$$

$$t=\ln\frac{7}{6}$$

The mass passes through the equilibrium position when $t=\ln\frac{7}{6}$.

(c) We need to find a particular solution.

$$x_p=Ate^{-t} \implies \dot{x}_p=Ae^{-t}(1-t) \implies \ddot{x}_p=Ae^{-t}(t-2)$$

$$2\ddot{x}_p+6\dot{x}_p+4x_p=2Ae^{-t}(t-2)+6Ae^{-t}(1-t)+4Ate^{-t}$$

$$2Ate^{-t}-4Ae^{-t}+6Ae^{-t}-6Ate^{-t}+4Ate^{-t}=2Ae^{-t}=4e^{-t} \implies A=2 \text{ and } x_p=2te^{-t}$$

$$x(t)=x_h(t)+x_p(t)=c_1e^{-2t}+c_2e^{-t}+2te^{-t}$$

$$x(0)=c_1+c_2=1$$

$$\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}-2te^{-t}+2e^{-t} \implies \dot{x}(0)=-2c_1-c_2+2=-8 \implies -2c_1-c_2=-10$$

$$c_1=\frac{\begin{vmatrix}1 & 1\\ -10 & -1\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -2 & -1\end{vmatrix}}=9 \qquad c_2=\frac{\begin{vmatrix}1 & 1\\ -2 & -10\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -2 & -1\end{vmatrix}}=-8$$

$$x(t)=9e^{-2t}-8e^{-t}+2te^{-t}$$

When $t=2$, the position of the mass is $x(2)=9e^{-4}-4e^{-2}$

(d) We need $b^2 - 4mk = 6^2 - 4m(4) = 36 - 16m < 0 \implies \frac{9}{4} < m$

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4. [2360/041724 (22 pts)] This problem will consider the following initial value problem $y'' + 25y = 100e^{5t},\ y(0) = -2,\ y'(0) = 3$.

(a) (6 pts) Convert the initial value problem into a system of a first order equations with appropriate initial conditions, writing the system in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$, if possible.

(b) (6 pts) Find the partial fraction decomposition of $\dfrac{100}{(s-5)(s^2+25)}$. Make sure you do this correctly; it will be useful in the next part.

(c) (10 pts) Use Laplace transforms to solve the initial value problem.

SOLUTION:

(a)

$$x_1 = y, \quad x_2 = y', \quad \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1' = y' = x_2, \quad x_2' = y'' = 100e^{5t} - 25x_1$$

$$\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 0 \\ 100e^{5t} \end{bmatrix}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

(b)

$$\frac{100}{(s-5)(s^2+25)} = \frac{A}{s-5} + \frac{Bs+C}{s^2+25}$$

$$100 = A\left(s^2+25\right) + (Bs+C)(s-5)$$

$$100 = (A+B)s^2 + (-5B+C)s + (25A-5C) \qquad \text{equate coefficients}$$

$$A + B = 0 \implies A = -B$$

$$-5B + C = 0 \implies C = 5B$$

$$25A - 5C = 100 \implies 5A - C = 20 \implies 5(-B) - 5B = 20 \implies B = -2 \implies A = 2, C = -10$$

$$\frac{100}{(s-5)(s^2+25)} = \frac{2}{s-5} - \frac{2s+10}{s^2+25}$$

(c)

$$s^2Y(s) - sy(0) - y'(0) + 25Y(s) = \frac{100}{s-5}$$

$$\left(s^2+25\right)Y(s) = \frac{100}{s-5} - 2s + 3$$

$$Y(s) = \frac{100}{(s-5)(s^2+25)} - \frac{2s}{s^2+25} + \frac{3}{s^2+25}$$

$$= \frac{2}{s-5} - \frac{2s+10}{s^2+25} - \frac{2s}{s^2+25} + \frac{3}{s^2+25}$$

$$= \frac{2}{s-5} - \frac{4s}{s^2+25} - \frac{7}{s^2+25}$$

$$y(t) = \mathscr{L}^{-1}\left\{\frac{2}{s-5} - \frac{4s}{s^2+25} - \frac{7}{5}\frac{5}{s^2+25}\right\}$$

$$= 2e^{5t} - 4\cos 5t - \frac{7}{5}\sin 5t$$

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5. [2360/041724 (20 pts) Consider the differential equation $y^{(6)} = 16y'' + f(t)$.

(a) (10 pts) Find a basis for the solution space when $f(t) = 0$.

(b) (10 pts) For each of the following functions $f(t)$, write down the form of the particular solution you would use to solve the nonhomogeneous equation using using the Method of Undetermined Coefficients. Do not find the constants and write N/A if the method is not applicable.

i. $f(t) = 3t(t-4)$ ii. $f(t) = te^t + e^{-2t}$ iii. $f(t) = \cos t + \sin 3t$ iv. $f(t) = t \sin 2t$ v. $f(t) = \dfrac{\sin 2t}{e^{2t}}$

<u>SOLUTION:</u>

(a) The homogeneous equation is $y^{(6)} - 16y'' = 0$ with characteristic equation

$$r^6 - 16r^2 = r^2\left(r^4 - 16\right) = r^2\left(r^2 - 4\right)\left(r^2 + 4\right) = r^2(r-2)(r+2)\left(r^2 + 4\right) = 0$$

$$\implies r = 0 \text{ (multiplicity 2)}, r = -2, r = 2, r = 2i, r = -2i$$

$$\text{basis for solution space is } \left\{1, t, e^{2t}, e^{-2t}, \cos 2t, \sin 2t\right\}$$

(b) i. $y_p = t^2(At^2 + Bt + C)$

ii. $y_p = (At + B)e^t + Cte^{-2t}$

iii. $y_p = A\cos t + B\sin t + C\cos 3t + D\sin 3t$

iv. $y_p = t\left[(At + B)\sin 2t + (Ct + D)\cos 2t\right]$

v. $y_p = Ae^{-2t}\sin 2t + Be^{-2t}\cos 2t$

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