- This exam is worth 100 points and has 5 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED <u>AND</u> UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5”× 11” crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: ”I will abide by the CU Boulder Honor Code on this exam.” **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/041724 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) $7\ddot{x} + 6\dot{x} + e^x + \cos x = 0$ describes a conservative system.

   (b) If y_1, y_2, y_3 are solutions to a third order linear, homogeneous differential equation, then the solution space must be equal to span $\{y_1, y_2, y_3\}$.

   (c) $\mathscr{L}\{t \cosh t\} = \dfrac{s^2+1}{(s^2-1)^2}$

   (d) If the characteristic equation of the differential equation governing a forced oscillator is $2r^2 + 2r + 1 = 0$, and the oscillator is forced by $f(t) = 10\cos\left(\frac{t}{2}\right)$, then the oscillator is in resonance.

   (e) If the charge in a circuit is given by $q(t) = e^{-4t}(\cos t + \sin t) + \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$, then the value of the steady state charge when $t = \frac{\pi}{4}$ is $\frac{1}{2}$.

2. [2360/041724 (24 pts)] Use variation of parameters to find the general solution of $t^2u'' + tu' - 9u = 72(t+1)$, assuming solutions of the associated homogeneous equation are of the form $u = t^r$. Be sure to simplify your final answer.

3. [2360/041724 (24 pts)] Consider the initial value problem $2\ddot{x} + 6\dot{x} + 4x = f(t), x(0) = 1, \dot{x}(0) = -8$ describing a certain mass/spring harmonic oscillator.

   (a) (2 pts) Is the oscillator underdamped, overdamped or critically damped? Justify your answer.

   (b) (10 pts) Assuming that oscillator is unforced/undriven, $f(t) = 0$, determine if and when the mass will pass through the equilibrium position.

   (c) (10 pts) Suppose now that the oscillator is forced by $f(t) = 4e^{-t}$. Using the same initial conditions, use the Method of Undetermined Coefficients to find the position of the mass when $t = 2$.

   (d) (2 pts) Now suppose you have the ability to change the value of the mass, m, in the mass/spring system. Find all the values of m, if any, that will allow the mass to pass through its equilibrium position more than once, assuming the damping and restoring constants are their original values.

4. [2360/041724 (22 pts)] This problem will consider the following initial value problem $y'' + 25y = 100e^{5t},\ y(0) = -2,\ y'(0) = 3$.

   (a) (6 pts) Convert the initial value problem into a system of a first order equations with appropriate initial conditions, writing the system in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$, if possible.

   (b) (6 pts) Find the partial fraction decomposition of $\dfrac{100}{(s-5)(s^2+25)}$. Make sure you do this correctly; it will be useful in the next part.

   (c) (10 pts) Use Laplace transforms to solve the initial value problem.

MORE PROBLEMS AND LAPLACE TABLE ON REVERSE

5. [2360/041724 (20 pts)] Consider the differential equation $y^{(6)} = 16y'' + f(t)$.

(a) (10 pts) Find a basis for the solution space when $f(t) = 0$.

(b) (10 pts) For each of the following functions $f(t)$, write down the form of the particular solution you would use to solve the nonhomogeneous equation using using the Method of Undetermined Coefficients. Do not find the constants and write N/A if the method is not applicable.

i. $f(t) = 3t(t-4)$ ii. $f(t) = te^{t} + e^{-2t}$ iii. $f(t) = \cos t + \sin 3t$ iv. $f(t) = t\sin 2t$ v. $f(t) = \dfrac{\sin 2t}{e^{2t}}$

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
