1. [2360/031324 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) All $n \times n$ matrices with 0 on and below the main diagonal are singular.

   (b) Suppose $\mathbf{Q}$ is a matrix with the property that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. If $\mathbf{H}$ is invertible and $\mathbf{H}\left(\mathbf{Q}^{\mathrm{T}}\mathbf{H}\right)^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$ then $\vec{\mathbf{x}} = \mathbf{Q}\vec{\mathbf{y}}$.

   (c) Given $\mathbf{C}\vec{\mathbf{y}} = \vec{\mathbf{w}}$ where $\mathbf{C}$ is a 5×5 matrix with eigenvalues $\{0, 0, 1, 3+5i, 3-5i\}$, Cramer's rule can be used to find the solution $\vec{\mathbf{y}}$.

   (d) If $\mathbf{C} = \begin{bmatrix} 2 & 0 \\ -8 & -2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$, then $\mathbf{CD} = \mathbf{DC}$.

   (e) Any set of 4 vectors in the vector space $\mathbb{R}^4$ will always span $\mathbb{R}^4$.

SOLUTION:

(a) **TRUE** The determinant of an upper triangular matrix is the product of the diagonal elements, which are all zero in this instance, making the determinant 0, implying that the matrix is singular.

(b) **FALSE**

$$\mathbf{H}\left(\mathbf{Q}^{\mathrm{T}}\mathbf{H}\right)^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$$
$$\mathbf{H}\mathbf{H}^{-1}\left(\mathbf{Q}^{\mathrm{T}}\right)^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$$
$$\mathbf{I}\left(\mathbf{Q}^{-1}\right)^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$$
$$\mathbf{Q}\vec{\mathbf{x}} = \vec{\mathbf{y}}$$
$$\vec{\mathbf{x}} = \mathbf{Q}^{\mathrm{T}}\vec{\mathbf{y}}$$

(c) **FALSE** Since 0 is an eigenvalue of $\mathbf{C}$, $|\mathbf{C}| = 0$, implying that $\mathbf{C}$ is singular, further implying that Cramer's Rule cannot be used.

(d) **TRUE** $\mathbf{CD} = \begin{bmatrix} 2 & 0 \\ -8 & -2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -28 & -6 \end{bmatrix}$ and $\mathbf{DC} = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -8 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -28 & -6 \end{bmatrix}$

(e) **FALSE** The four vectors must be linearly independent to span $\mathbb{R}^4$.

■

2. [2360/031324 (12 pts)] Find all values of k, if any, for which the vector $\begin{bmatrix} 1 \\ k \\ 2 \end{bmatrix}$ is in the column space of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 3 & 6 & 4 & 1 & 1 \end{bmatrix}$

SOLUTION:

This is equivalent to finding values of k that make $\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ k \\ 2 \end{bmatrix}$ consistent.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 & k \\ 3 & 6 & 4 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 & k \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & k+2 \end{array}\right]$$

Thus, if $k = -2$, $\begin{bmatrix} 1 \\ k \\ 2 \end{bmatrix} \in \operatorname{Col}\mathbf{A}$. ■

3. [2360/031324 (20 pts)] After a number of elementary row operations, the augmented matrix associated with a certain linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{array}\right]$$

(a) (2 pts) Is the matrix in RREF? If not, find the RREF.

(b) (2 pts) How many solutions are there to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Briefly justify your answer.

(c) (2 pts) What is $|\mathbf{A}|$? Briefly justify your answer.

(d) (2 pts) What is the rank of $\mathbf{A}$? Briefly justify your answer.

(e) (12 pts) The following parts concern the associated homogeneous system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$?

i. (2 pts) Write down the RREF of the augmented matrix for this system.

ii. (2 pts) How many solutions does the system have?

iii. (6 pts) Find a basis for the solution space.

iv. (2 pts) What is the dimension of the solution space?

SOLUTION:

(a) No. The RREF is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(b) None. The system is inconsistent. This follows from the fact that one row of the RREF is $0\ 0\ 0\,|\,1$.

(c) $|\mathbf{A}| = 0$. The coefficient matrix contains a row of all zeros.

(d) rank $\mathbf{A} = 2$. There are two pivot columns in the RREF.

(e) i. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

ii. infinitely many

iii. Solutions are of the form $\vec{\mathbf{x}} = \begin{bmatrix} -r \\ -3r \\ r \end{bmatrix}$, $r \in \mathbb{R}$. A basis for the solution space is therefore $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$.

iv. The dimension of the solution space is 1.

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4. [2360/031324 (15 pts)] Use the inverse of an appropriate matrix to find the solution of the system
$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= -2 \\ x_1 + 3x_2 + 4x_3 &= 2 \\ x_1 + 4x_2 + 3x_3 &= -1 \end{aligned}.$$

SOLUTION:

The system can be written as $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ with solution $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$. We find $\mathbf{A}^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3^*=-R_1+R_3]{R_2^*=-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{R_1^*=-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1^*=-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \implies \mathbf{A}^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\implies \vec{\mathbf{x}} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -17 \\ 1 \\ 4 \end{bmatrix}$$

■

5. [2360/031324 (15 pts) Determine if the following subsets, $\mathbb{W}$, of the given vector space, $\mathbb{V}$, are subspaces. Assume the standard operations appropriate to $\mathbb{V}$ apply. Justify your answers.

(a) (5 pts) $\mathbb{V} = \mathbb{R}^2$; $\mathbb{W} = \left\{(x_1, x_2) \in \mathbb{R}^2 \,\middle|\, |x_1| = |x_2|\right\}$

(b) (5 pts) $\mathbb{V} = \mathbb{P}_1$; $\mathbb{W}$ is the set of all polynomials having the form $\frac{m}{n} + at$, where a is a real number and m, n are integers.

(c) (5 pts) $\mathbb{V} = \mathbb{M}_{22}$; $\mathbb{W}$ is the set of all 2×2 matrices whose off-diagonal elements sum to zero.

SOLUTION:

(a) Not a subspace. Not closed under vector addition: $(1, -1)$ and $(1, 1)$ are in $\mathbb{W}$, but $(1, -1) + (1, 1) = (2, 0)$ is not in $\mathbb{W}$.

(b) Not a subspace. Not closed under scalar multiplication. For example, $\frac{3}{4} + t \in \mathbb{W}$ but $\sqrt{3}\left(\frac{3}{4} + t\right) = \frac{3\sqrt{3}}{4} + \sqrt{3}t \notin \mathbb{W}$ since $3\sqrt{3}/4$ is not a ratio of integers.

(c) Subspace. $\mathbb{W}$ is nonempty. Let $\vec{\mathbf{u}} = \begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} a_2 & b_2 \\ -b_2 & c_2 \end{bmatrix}$ be in $\mathbb{W}$ and let $p, q \in \mathbb{R}$. Then

$$p\vec{\mathbf{u}} + q\vec{\mathbf{v}} = p\begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} + q\begin{bmatrix} a_2 & b_2 \\ -b_2 & c_2 \end{bmatrix} = \begin{bmatrix} pa_1 & pb_1 \\ -pb_1 & pc_1 \end{bmatrix} + \begin{bmatrix} qa_2 & qb_2 \\ -qb_2 & qc_2 \end{bmatrix} = \begin{bmatrix} pa_1 + qa_2 & pb_1 + qb_2 \\ -(pb_1 + qb_2) & pc_1 + pc_2 \end{bmatrix} \in \mathbb{W}$$

■

6. [2360/031324 (12 pts)] Let $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$.

(a) (5 pts) Find the eigenvalues of $\mathbf{A}$.

(b) (7 pts) Find the eigenvector associated with the eigenvalue from part (a) with positive imaginary part.

SOLUTION:

(a)

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{vmatrix} = (4-\lambda)(-\lambda) + 5 = \lambda^2 - 4\lambda + 5 = 0 \implies \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(5)}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

(b) Need to solve $[\mathbf{A} - (2+i)\mathbf{I}]\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$

$$\left[\begin{array}{cc|c} 4-(2+i) & -5 & 0 \\ 1 & -(2+i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2-i & 0 \\ 2-i & -5 & 0 \end{array}\right] \xrightarrow{R_2^* = (-2+i)R_1 + R_2} \left[\begin{array}{cc|c} 1 & -2-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The first row of the matrix yields $v_1 - (2+i)v_2 = 0 \implies v_1 = (2+i)v_2 \implies \vec{\mathbf{v}} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

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7. [2360/031324 (16 pts)] Consider the set of vectors in $\mathbb{P}_2$ given by $\{1 - 2x^2, 2x - 1, x^2 + x - 1\}$.

(a) (8 pts) Show that the Wronskian of the functions is inconclusive when deciding whether or not the set is linearly independent.

(b) (8 pts) Does the set form a basis for $\mathbb{P}_2$? Justify your answer.

SOLUTION:

(a)

$$W(x) = \begin{vmatrix} 1-2x^2 & 2x-1 & x^2+x-1 \\ -4x & 2 & 2x+1 \\ -4 & 0 & 2 \end{vmatrix} = -4(-1)^{3+1}\begin{vmatrix} 2x-1 & x^2+x-1 \\ 2 & 2x+1 \end{vmatrix} + 2(-1)^{3+3}\begin{vmatrix} 1-2x^2 & 2x-1 \\ -4x & 2 \end{vmatrix}$$

$$= -4\left(4x^2 + 2x - 2x - 1 - 2x^2 - 2x + 2\right) + 2\left(2 - 4x^2 + 8x^2 - 4x\right)$$

$$= -8x^2 + 8x - 4 + 4 + 8x^2 - 8x = 0$$

Since the Wronskian is identically 0, we cannot determine if the functions are linearly independent.

(b) Since there are 3 vectors in a vector space of dimension 3, they will form a basis if they are linearly independent. Since the Wronskian test for linear independence was inconclusive, we check to see if the only solution to

$$c_1\left(1-2x^2\right)+c_2\left(2x-1\right)+c_3\left(x^2+x-1\right)=0+0x+0x^2$$

is the trivial solution, $c_1=c_2=c_3=0$. Equating coefficients gives

$$\left.\begin{aligned} c_1-c_2-c_3&=0\\ 2c_2+c_3&=0\\ -2c_1+c_3&=0 \end{aligned}\right\}\implies \begin{bmatrix}1&-1&-1\\0&2&1\\-2&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{vmatrix}1&-1&-1\\0&2&1\\-2&0&1\end{vmatrix}=1(-1)^{1+1}\begin{vmatrix}2&1\\0&1\end{vmatrix}-2(-1)^{3+1}\begin{vmatrix}-1&-1\\2&1\end{vmatrix}=(1)(2)-2(1)=0$$

indicating that the system has nontrivial solutions and the vectors are linearly dependent. Consequently, they cannot form a basis for $\mathbb{P}_2$.

Alternatively, one might, by inspection, be able to see that $1(1-2x^2)-1(2x-1)+2(x^2+x-1)=0+0x+0x^2$, also indicating that the vectors are linearly dependent.

As another alternative, the RREF is $\left[\begin{array}{ccc|c}1&0&-\frac{1}{2}&0\\0&1&\frac{1}{2}&0\\0&0&0&0\end{array}\right]$, indicating the existence of nontrivial solutions.

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