- This exam is worth 100 points and has 7 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"$\times$ 11" crib sheet with writing on one side.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/031324 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) All $n \times n$ matrices with 0 on and below the main diagonal are singular.

   (b) Suppose $\mathbf{Q}$ is a matrix with the property that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. If $\mathbf{H}$ is invertible and $\mathbf{H}\left(\mathbf{Q}^{\mathrm{T}}\mathbf{H}\right)^{-1}\vec{\mathbf{x}} = \vec{\mathbf{y}}$ then $\vec{\mathbf{x}} = \mathbf{Q}\vec{\mathbf{y}}$.

   (c) Given $\mathbf{C}\vec{\mathbf{y}} = \vec{\mathbf{w}}$ where $\mathbf{C}$ is a 5×5 matrix with eigenvalues $\{0, 0, 1, 3 + 5i, 3 - 5i\}$, Cramer's rule can be used to find the solution $\vec{\mathbf{y}}$.

   (d) If $\mathbf{C} = \begin{bmatrix} 2 & 0 \\ -8 & -2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$, then $\mathbf{CD} = \mathbf{DC}$.

   (e) Any set of 4 vectors in the vector space $\mathbb{R}^4$ will always span $\mathbb{R}^4$.

2. [2360/031324 (12 pts)] Find all values of k, if any, for which the vector $\begin{bmatrix} 1 \\ k \\ 2 \end{bmatrix}$ is in the column space of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 3 & 6 & 4 & 1 & 1 \end{bmatrix}$

3. [2360/031324 (20 pts)] After a number of elementary row operations, the augmented matrix associated with a certain linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{array}\right]$$

   (a) (2 pts) Is the matrix in RREF? If not, find the RREF.

   (b) (2 pts) How many solutions are there to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Briefly justify your answer.

   (c) (2 pts) What is $|\mathbf{A}|$? Briefly justify your answer.

   (d) (2 pts) What is the rank of $\mathbf{A}$? Briefly justify your answer.

   (e) (12 pts) The following parts concern the associated homogeneous system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$?

      i. (2 pts) Write down the RREF of the augmented matrix for this system.

      ii. (2 pts) How many solutions does the system have?

      iii. (6 pts) Find a basis for the solution space.

      iv. (2 pts) What is the dimension of the solution space?

4. [2360/031324 (15 pts)] Use the inverse of an appropriate matrix to find the solution of the system
$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= -2 \\ x_1 + 3x_2 + 4x_3 &= 2 \\ x_1 + 4x_2 + 3x_3 &= -1 \end{aligned}.$$

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5. [2360/031324 (15 pts) Determine if the following subsets, $\mathbb{W}$, of the given vector space, $\mathbb{V}$, are subspaces. Assume the standard operations appropriate to $\mathbb{V}$ apply. Justify your answers.

   (a) (5 pts) $\mathbb{V} = \mathbb{R}^2; \mathbb{W} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; |x_1| = |x_2| \right\}$

   (b) (5 pts) $\mathbb{V} = \mathbb{P}_1; \mathbb{W}$ is the set of all polynomials having the form $\frac{m}{n} + at$, where a is a real number and m, n are integers.

   (c) (5 pts) $\mathbb{V} = \mathbb{M}_{22}; \mathbb{W}$ is the set of all 2×2 matrices whose off-diagonal elements sum to zero.

6. [2360/031324 (12 pts)] Let $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$.

   (a) (5 pts) Find the eigenvalues of $\mathbf{A}$.

   (b) (7 pts) Find the eigenvector associated with the eigenvalue from part (a) with positive imaginary part.

7. [2360/031324 (16 pts)] Consider the set of vectors in $\mathbb{P}_2$ given by $\left\{1 - 2x^2, 2x - 1, x^2 + x - 1\right\}$.

   (a) (8 pts) Show that the Wronskian of the functions is inconclusive when deciding whether or not the set is linearly independent.

   (b) (8 pts) Does the set form a basis for $\mathbb{P}_2$? Justify your answer.