1. [2360/021424 (10 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The differential equation $x'(t) = -2(t-1)(x+1)^2$ has two equilibrium solutions.

   (b) Suppose L is a linear operator and $y(t), z(t)$ and $w(t)$ are functions such that $L[y(t)] = 3e^{-t}, L[z(t)] = 0$ and $L[w(t)] = -\sin t$. Then $L[y(t) - 3w(t) + 2z(t)] = 3\left(e^{-t} + \sin t\right)$.

   (c) $w(x) = xe^x$ is a solution to $xw' - w = x^2e^x$ on the entire real line.

   (d) Picard's theorem can be used to show that the initial value problem $y' = \sqrt[3]{ty},\ y(1) = 0$ has a nonunique solution.

   (e) The isoclines of the equation $y' - 2(t-y) + 1 = 0$ are a family of lines all having the same slope.

   SOLUTION:

   (a) **FALSE** $x = -1$ is the sole equilibrium solution.

   (b) **TRUE** $L[y(t) - 3w(t) + 2z(t)] = L[y(t)] - 3L[w(t)] + 2L[z(t)] = 3e^{-t} - 3(-\sin t) + 2(0) = 3\left(e^{-t} + \sin t\right)$

   (c) **FALSE** The function is a solution, $xw' - w = x(xe^x + e^x) - xe^x = x^2e^x$, but not on any interval containing 0, thus not on the entire real line.

   (d) **FALSE** $f(t,y) = t^{1/3}y^{1/3}$ and $f_y(t,y) = \frac{1}{3}t^{1/3}y^{-2/3}$. Since f_y is not defined when $y = 0$, it is not continuous in any rectangle containing $(1,0)$ and Picard's theorem tells us nothing about the uniqueness of solutions to the IVP (the solution may be unique, it may not be).

   (e) **TRUE** $y' = 2(t-y) - 1$ implies that the isoclines are the lines $2(t-y) - 1 = c$ or $y = t - \frac{c+1}{2}$ which all have slope of 1.

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2. [2360/021424 18 pts)] Use the integrating factor method to solve the following initial value problem, identifying the transient and steady state solutions, if any exist.

$$\frac{t}{2}Q' + Q = \frac{\cos 2t}{t} + 3,\ Q(\pi) = 4,\ t > 0$$

   SOLUTION:

$$Q' + \frac{2}{t}Q = \frac{2\cos 2t}{t^2} + \frac{6}{t}$$

$$\mu(t) = e^{\int \frac{2}{t}\,\mathrm{d}t} = t^2$$

$$\int \left(t^2Q\right)'\,\mathrm{d}t = \int (2\cos 2t + 6t)\,\mathrm{d}t$$

$$t^2Q = \sin 2t + 3t^2 + C$$

$$Q(t) = \frac{\sin 2t}{t^2} + 3 + \frac{C}{t^2}$$

$$Q(\pi) = \frac{\sin 2\pi}{\pi^2} + 3 + \frac{C}{\pi^2} = 4 \implies C = \pi^2$$

$$Q(t) = \underbrace{\frac{\sin 2t + \pi^2}{t^2}}_{\text{transient}} + \underbrace{3}_{\text{steady state}}$$

   ■

3. [2360/021424 (19 pts)] The population of a certain species is given by $S(t)$, where t is measured in years and $S(t)$ is measured in hundreds of individuals, that is, $S = 2$ implies that there are 200 individuals present. The evolution of the population is governed by the equation $S' = 2tS - 4t$ and there are 100 individuals initially. Use the Euler-Lagrange two stage method (variation of parameters) to determine if the species will go extinct in a finite amount of time, t_f. Find t_f or explain why the species does not go extinct.

   SOLUTION:

   Stage 1: Solve the associated homogeneous problem using separation of variables.

$$\int \frac{\mathrm{d}S_h}{S_h} = \int 2t\,\mathrm{d}t$$

$$\ln|S_h| = t^2 + k$$

$$S_h(t) = Ce^{t^2}$$

Stage 2: Find a particular solution.

$$S_p(t) = v(t)e^{t^2}$$

$$S_p' - 2tS_p = 2tve^{t^2} + v'e^{t^2} - 2tve^{t^2} = v'e^{t^2} = -4t$$

$$v(t) = \int v'(t)\,\mathrm{d}t = \int -4te^{-t^2}\,\mathrm{d}t = 2e^{-t^2}$$

$$S_p(t) = 2e^{t^2}e^{-t^2} = 2$$

Apply the Nonhomogeneous Principle and the initial condition.

$$S(t) = S_h(t) + S_p(t) = Ce^{t^2} + 2$$

$$S(0) = 1 = C + 2 \implies C = -1$$

$$S(t) = 2 - e^{t^2}$$

To see if the species will go extinct, we need to solve

$$2 - e^{t_f^2} = 0$$

$$t_f^2 = \ln 2$$

$$t_f = \sqrt{\ln 2}$$

The species goes extinct in $\sqrt{\ln 2}$ years. ■

4. [2360/021424 (18 pts)] Euler-homogeneous equations, studied in the homework, are nonseparable equations of the form $y' = f(y/x)$. Recall that these can be made separable by a change of variable, $v = y/x$. Use this information to find the explicit general solution of $2xyy' - x^2 - 3y^2 = 0$.

SOLUTION:

Rewrite the differential equation as $y' = \dfrac{x^2 + 3y^2}{2xy} = \dfrac{1}{2}\left(\dfrac{x}{y} + 3\dfrac{y}{x}\right)$.

$$v = y/x \implies y = vx \implies y' = v + xv'$$

$$v + xv' = \frac{1}{2}\left(\frac{1}{v} + 3v\right)$$

$$xv' = \frac{1}{2}\left(\frac{1}{v} + v\right)$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}\left(\frac{1 + v^2}{v}\right)$$

$$\int \frac{v}{1 + v^2}\,\mathrm{d}v = \frac{1}{2}\int \frac{\mathrm{d}x}{x}$$

$$\frac{1}{2}\ln\left|1 + v^2\right| = \frac{1}{2}\ln|x| + C$$

$$\ln\left|1 + \left(\frac{y}{x}\right)^2\right| - \ln|x| = C$$

$$\ln\left|\frac{x^2 + y^2}{x^3}\right| = C$$

$$\frac{x^2 + y^2}{x^3} = C$$

$$y = \pm\sqrt{Cx^3 - x^2}$$

■

5. [2360/021424 (17 pts) Consider the differential equation $w' = w^4 - 9w^2$.

(a) (5 pts) Suppose one step of Euler's Method is applied to the initial value problem consisting of the differential equation and the initial condition $w(0) = w_1 = 1$, yielding the approximation $w_2 = \frac{1}{2}$. What stepsize was used to compute this approximation?

(b) (10 pts) Find all of the equilibrium solutions and determine their stability. Plot the phase line.

(c) (2 pts) For what initial values of w will solutions be bounded as $t \to \infty$?

SOLUTION:

(a) Euler's method for this equation is $w_{n+1} = w_n + h(w_n^4 - 9w_n^2)$. This gives

$$w_2 = w_1 + h(w_1^4 - 9w_1^2) \implies \frac{1}{2} = 1 + h\left[1^4 - 9(1)^2\right] \implies -\frac{1}{2} = h(-8) \implies h = \frac{1}{16}$$

(b)

$$w^4 - 9w^2 = w^2\left(w^2 - 9\right) = w^2(w-3)(w+3) \implies \text{equilibrium solutions are } w = -3, 0, 3$$

$$3 < w \implies w' > 0$$

$$0 < w < 3 \implies w' < 0$$

$$-3 < w < 0 \implies w' < 0$$

$$w < -3 \implies w' > 0$$

$w = 3$ is unstable, $w = 0$ is semistable, $w = -3$ is stable

↑
3 ○
↓
0 ◓
↓
−3 ●
↑

(c) $w \leq 3$

■

6. [2360/021424 (18 pts)] Consider the following Lotka-Volterra predator-prey equations, where x represents the prey and y the predator.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 100x - 20xy$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -60y + 20xy$$

(a) (4 pts) Find the h nullcline(s).

(b) (4 pts) Find the v nullcline(s).

(c) (4 pts) Find all the equilibrium points, if any exist.

(d) (6 pts) Determine if the predator and prey populations are increasing, decreasing or remaining constant at the following points:

i. $(5, 5)$ ii. $(3, 10)$ iii. $(1, 2)$

SOLUTION:

(a) h nullclines are where $\mathrm{d}y/\mathrm{d}t = 0$, that is,

$$-60y + 20xy = 20y(-3 + x) = 0 \implies y = 0, x = 3$$

(b) v nullclines are where $\mathrm{d}x/\mathrm{d}t = 0$, that is,

$$100x - 20xy = 20x(5 - y) = 0 \implies x = 0, y = 5$$

(c) Equilibrium solutions occur where the h and v nullclines intersect. This happens at $(0, 0)$ and $(3, 5)$. Note also that if $y = 0$ in $20y(-3 + x) = 0$, then $x = 0$ in $20x(5 - y) = 0$. And, if $x = 3$ in $20y(-3 + x) = 0$, then $y = 5$ in $20x(5 - y) = 0$. This is another approach to finding the equilibrium points.

(d)

i. $(5,5)$

$$\mathrm{d}x/\mathrm{d}t = (100)(5) - 20(5)(5) = 0 \implies \text{prey constant; } \mathrm{d}y/\mathrm{d}t = (-60)(5) + 20(5)(5) > 0 \implies \text{predator increasing}$$

ii. $(3,10)$

$$\mathrm{d}x/\mathrm{d}t = (100)(3) - 20(3)(10) < 0 \implies \text{prey decreasing; } \mathrm{d}y/\mathrm{d}t = (-60)(10) + 20(3)(10) = 0 \implies \text{predator constant}$$

iii. $(1,2)$

$$\mathrm{d}x/\mathrm{d}t = (100)(1) - 20(1)(2) > 0 \implies \text{prey increasing; } \mathrm{d}y/\mathrm{d}t = (-60)(2) + 20(1)(2) < 0 \implies \text{predator decreasing}$$

■