1. [2360/050823 (18 pts)] Consider the initial value problem $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -2\left(\theta - \theta_s\right),\ \theta(0) = \theta_0$ where θ_s is a constant.

   (a) (8 pts) Solve the initial value problem using the integrating factor method.

   (b) (4 pts) Verify that your answer to part (a) is the solution to the initial value problem.

   (c) (6 pts) Suppose the initial value problem models the rate of change of the temperature, θ, of an object immersed in a water bath with constant temperature, θ_s. Suppose the initial temperature of the object, $\theta_0 = 13°$C, and after $t = \ln 2$ minutes, the object's temperature is $10°$C. What is the temperature of the water bath, θ_s?

SOLUTION:

(a) Rewrite the equation as $\frac{\mathrm{d}\theta}{\mathrm{d}t} + 2\theta = 2\theta_s$. The integrating factor is then $\mu(t) = e^{2t}$. We then have

$$\begin{aligned}
\left(e^{2t}\theta\right)' &= 2\theta_s e^{2t} \\
e^{2t}\theta &= \int 2\theta_s e^{2t}\,\mathrm{d}t = \theta_s e^{2t} + C \\
\theta &= \theta_s + Ce^{-2t} \qquad \text{apply the initial condition} \\
\theta_0 &= \theta_s + C(1) \implies C = \theta_0 - \theta_s \\
\theta(t) &= \theta_s + \left(\theta_0 - \theta_s\right)e^{-2t}
\end{aligned}$$

(b) Substitute the solution into the IVP.

$$\theta(0) = \theta_s + \left(\theta_0 - \theta_s\right)e^{2(0)} = \theta_s + \theta_0 - \theta_s = \theta_0 \qquad \text{initial condition satisfied}$$

$$\left.\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -2\left(\theta_0 - \theta_s\right)e^{-2t} \\
-2\left(\theta - \theta_s\right) &= -2\left[\theta_s + \left(\theta_0 - \theta_s\right)e^{-2t} - \theta_s\right] = -2\left(\theta_0 - \theta_s\right)e^{-2t}
\end{aligned}\right\}\ \text{equal}$$

(c)

$$\begin{aligned}
10 &= \theta_s + \left(13 - \theta_s\right)e^{-2\ln 2} \\
10 &= \theta_s + \left(13 - \theta_s\right)\left(\frac{1}{4}\right) \\
10 - \frac{13}{4} &= \frac{3}{4}\theta_s \\
\frac{4}{3}\left(\frac{40-13}{4}\right) &= \theta_s = 9°\text{C}
\end{aligned}$$

■

2. [2360/050823 (16 pts)] On a separate page in your bluebook, write the letters (a) through (h) in a column. Then for the following questions, write the word **TRUE** or **FALSE** next to each letter, as appropriate. No partial credit given and no work need be shown. If you do any work to come up with your answers, please do it elsewhere - do not include it in your list of answers (this helps with grading).

   (a) If $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{u}} \in \operatorname{span}\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$, then there exist c_1, c_2, c_3, not all zero, such that $c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}} = \vec{\mathbf{0}}$.

   (b) If $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 4 & 2 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, then $\mathbf{A}^{\mathrm{T}}\left(\mathbf{B} - 2\mathbf{I}\right) = \mathbf{I}$.

   (c) If $|\mathbf{AB}| = 3$, where $\mathbf{A}$ and $\mathbf{B}$ are both square matrices, then $\mathbf{A}$ must be invertible.

   (d) $y = 5$ is a stable equilibrium solution of $y' = \left(y^2 - 4y + 3\right)\left(y - 5\right)^2$.

   (e) The nullclines of any linear system of differential equations of the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$, where $\mathbf{A}$ is a 2×2 matrix with constant entries (not all zero), are lines through the origin.

   (f) Cramer's Rule can be used to solve a system of three equations in three unknowns with lower triangular coefficient matrix whose diagonal entries are $-1, 0, 2$.

   (g) Picard's Theorem can be used to determine if the initial value problem $y' = \sqrt[3]{t(y-1)},\ y(0) = 1$ has a unique solution.

(h) $x' = e^{x-\sin t}\cos t$ can be solved using separation of variables.

SOLUTION:

(a) **TRUE** $\vec{\mathbf{u}} \in \text{span}\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\} \implies \vec{\mathbf{u}} = c_1\vec{\mathbf{v}} + c_2\vec{\mathbf{w}}$ where c_1, c_2 are not both zero (since $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$). In other words, the vectors are linearly dependent.

(b) **TRUE**

$$\mathbf{A}^{\mathrm{T}}(\mathbf{B} - 2\mathbf{I}) = \mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 5 & 4 & 2 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

(c) **TRUE** $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = 3 \implies |\mathbf{A}| \neq 0 \implies$ **A** is invertible

(d) **FALSE** $y' = (y-3)(y-1)(y-5)^2$. If $y > 5, y' > 0$. If $3 < y < 5, y' > 0 \implies y = 5$ is a semistable equlibrium solution.

(e) **TRUE** If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the v nullcline is $ax_1 + bx_2 = 0$ and the h nullcline is $cx_1 + dx_2 = 0$, both lines through the origin.

(f) **FALSE** The determinant of the coefficient matrix in this case is the product of the diagonal elements $(-1)(0)(2) = 0$.

(g) **FALSE** $f(t, y) = \sqrt[3]{t(y-1)}, f_y(t, y) = \frac{1}{3}t^{1/3}(y-1)^{-2/3}$. The latter function is not defined at $(0, 1)$ and is therefore not continuous at $(0, 1)$ nor in any rectangle containing $(0, 1)$. The theorem guarantees a solution but we cannot conclude from the theorem that the solution is unique.

(h) **TRUE** The equation can be separated thus: $e^{-x}\,\mathrm{d}x = e^{-\sin t}\cos t\,\mathrm{d}t$.

■

3. [2360/050823 (20 pts)] The following parts are not related.

(a) (10 pts) Find the general solution of $x^2w'' - 2w = 3 - \dfrac{1}{x^2}$ assuming $x > 0$ and solutions to the homogeneous equation have the form $w = x^r$.

(b) (10 pts) Consider the differential equation $7u'' + 8u' + u = f(t)$. For each $f(t)$ below, give the form only (**do not solve** for the coefficients) of the particular solution guess that would be used in the Method of Undetermined Coefficients. If that method is not possible, write NONE.

i. $f(t) = t\ln t$ ii. $f(t) = 1$ iii. $f(t) = 5e^{-t/7} + 7e^{-t}$ iv. $f(t) = \cos t - 13\sin 13t$ v. $f(t) = te^{-t}$

SOLUTION:

(a) Solve the associated homogeneous problem assuming solutions of the form $w = x^r$.

$$x^2w'' - 2w = x^2r(r-1)x^{r-2} - 2x^r = x^r(r^2 - r - 2) = 0 \implies r = -1, 2 \implies w_1 = x^{-1}, w_2 = x^2$$

Use variation of parameters to solve the nonhomogeneous problem, rewritten as $w'' - \dfrac{2}{x^2}w = 3x^{-2} - x^{-4}$ assuming a particular solution of the form $w_p(x) = v_1(x)x^{-1} + v_2(x)x^2$.

$$W[x^{-1}, x^2](x) = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3$$

$$v_1(x) = \int \frac{-x^2\left(3x^{-2} - x^{-4}\right)}{3}\,\mathrm{d}x = \frac{1}{3}\int\left(-3 + x^{-2}\right)\mathrm{d}x = -x - \frac{1}{3}x^{-1}$$

$$v_2(x) = \int \frac{x^{-1}\left(3x^{-2} - x^{-4}\right)}{3}\,\mathrm{d}x = \frac{1}{3}\int\left(3x^{-3} - x^{-5}\right)\mathrm{d}x = -\frac{1}{2}x^{-2} + \frac{1}{12}x^{-4}$$

$$w_p(x) = \left(-x - \frac{1}{3}x^{-1}\right)x^{-1} + \left(-\frac{1}{2}x^{-2} + \frac{1}{12}x^{-4}\right)x^2 = -1 - \frac{1}{3}x^{-2} - \frac{1}{2} + \frac{1}{12}x^{-2} = -\frac{3}{2} - \frac{1}{4}x^{-2}$$

Now use the Nonhomogeneous Principle to find the general solution as $w(x) = w_h(x) + w_p(x) = c_1x^{-1} + c_2x^2 - \dfrac{1}{4}x^{-2} - \dfrac{3}{2}$.

(b) We need to find the solutions to the associated homogeneous problem.

$$7r^2 + 8r + 1 = 0 \implies r = \frac{-8 \pm \sqrt{64 - (4)(7)(1)}}{(2)(7)} = \frac{-8 \pm 6}{14} = -1, -\frac{1}{7} \implies u_1 = e^{-t}, u_2 = e^{-t/7}$$

i. NONE
ii. $u_p = A$
iii. $u_p = t\left(Ae^{-t/7} + Be^{-t}\right)$
iv. $u_p = A\cos t + B\sin t + C\cos 13t + D\sin 13t$
v. $u_p = t(At+B)e^{-t}$

■

4. [2360/050823 (10 pts)] Two 10-liter tanks are initially completely filled with a well-mixed sugar solution. At time $t = 0$, two grams of sugar are dissolved in Tank 1 and one gram of sugar in Tank 2. For $t > 0$, water containing 4 grams of sugar per liter flows into Tank 1 at 3 liters per minute from an outside source. The sweet solution exits the tank system out of Tank 2 at 3 liters per minute. Furthermore, solution from Tank 2 flows into Tank 1 at 4 liters per minute and from Tank 1 into Tank 2 at 7 liters per minute.

(a) (8 pts) Set up, but **DO NOT** solve, an initial value problem that models the physical situation, writing your final answer using matrices and vectors.

(b) (2 pts) Without solving the system, over what time interval will the solution be valid?

SOLUTION:

(a) Let $x_1(t), x_2(t)$ represent the amount of sugar in Tank 1 and Tank 2, respectively. Then

$$x_1' = (3)(4) + 4\left(\frac{x_2}{10}\right) - 7\left(\frac{x_1}{10}\right) = -\frac{7}{10}x_1 + \frac{2}{5}x_2 + 12$$

$$x_2' = 7\left(\frac{x_1}{10}\right) - 7\left(\frac{x_2}{10}\right) = \frac{7}{10}x_1 - \frac{7}{10}x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -\frac{7}{10} & \frac{2}{5} \\ \frac{7}{10} & -\frac{7}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(b) Since the tanks' volumes never change the solution is valid for $t \in [0, \infty)$.

■

5. [2360/050823 (10 pts)] Consider the system $\begin{aligned} x_1 + 6x_2 &= 4 \\ 2x_1 + 4x_2 &= -8 \\ -x_1 + 2x_2 &= k \end{aligned}$. Find the value of k that makes the system consistent and find the corresponding solution for that value of k.

SOLUTION:

$$\left[\begin{array}{rr|r} 1 & 6 & 4 \\ 2 & 4 & -8 \\ -1 & 2 & k \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 6 & 4 \\ 0 & -8 & -16 \\ 0 & 8 & 4+k \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 6 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & k-12 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & k-12 \end{array}\right]$$

To be consistent, $k = 12$ in which case the unique solution is $\begin{bmatrix} -8 \\ 2 \end{bmatrix}$. ■

6. [2360/050823 (25 pts)] You have just finished setting up a mass/spring system on a table in your dorm room. The mass is 1 kg and the spring constant is 13 N/m. You have set things up so that the damping constant is 6 N/m/s.

(a) (15 pts) You have carefully adjusted the oscillator so that the mass itself is at rest at its equilibrium position, you start your stopwatch at $t = 0$ and you are ready to run your experiments. Unfortunately, as you walk away from the table, you bump into it, giving an impulsive force of 10 Newtons to the apparatus when $t = \pi/4$. Where is the mass with respect to its equilibrium position when $t = \pi$?

(b) (2 pts) After your mishap, you have reset everything as it was originally set up. You now plan on applying a force to the table on which the oscillator lies of $f(t) = \cos\sqrt{13}t$. Once this forcing occurs, will the oscillator exhibit beats, resonance, both or neither? Explain briefly.

(c) (8 pts) With your experiments done for the day, you go to bed and overnight your roommate sabotages the oscillator, changing out the spring to one whose spring constant is 9 N/m. Upon arising in the morning, you begin your experiments on the unforced oscillator, hoping to see it oscillate nicely. At $t = 0$, you push the mass 5 m to the left of the equilibrium and impart to it a 25 m/s velocity to the right. You know that something is wrong since you see the mass pass through the equilibrium position only once at $t = t_0$. Find the value of t_0.

SOLUTION:

(a) Let $x(t)$ be the displacement of the mass from its equilibrium position. The initial value problem that needs to be solved is then

$$\ddot{x} + 6\dot{x} + 13x = 10\delta\left(t - \frac{\pi}{4}\right),\ x(0) = 0,\ \dot{x}(0) = 0$$

$$\mathscr{L}\left\{\ddot{x} + 6\dot{x} + 13x = 10\delta\left(t - \frac{\pi}{4}\right)\right\}$$

$$s^2X(s) - sx(0) - \dot{x}(0) + 6\left[sX(s) - x(0)\right] + 13X(s) = 10e^{-\frac{\pi}{4}s}$$

$$\left(s^2 + 6s + 13\right)X(s) = 10e^{-\frac{\pi}{4}s}$$

$$X(s) = \frac{10e^{-\frac{\pi}{4}s}}{(s^2 + 6s + 13)} = 5e^{-\frac{\pi}{4}s}\left[\frac{2}{(s+3)^2 + 4}\right]$$

$$x(t) = \mathscr{L}^{-1}\left\{5e^{-\frac{\pi}{4}s}\left[\frac{2}{(s+3)^2 + 4}\right]\right\}$$

$$x(t) = 5e^{-3\left(t - \frac{\pi}{4}\right)}\sin\left[2\left(t - \frac{\pi}{4}\right)\right]\text{step}\left(t - \frac{\pi}{4}\right)$$

$$x(\pi) = 5e^{-3\left(\pi - \frac{\pi}{4}\right)}\sin\left[2\left(\pi - \frac{\pi}{4}\right)\right]\text{step}\left(\pi - \frac{\pi}{4}\right) = 5e^{-9\pi/4}\sin\left[2\left(3\pi/4\right)\right](1) = -5e^{-9\pi/4}$$

The mass is $5e^{-9\pi/4}$ m to the left of its equilibrium position.

(b) Since the oscillator is damped, neither beats nor resonance will occur, even though the forcing frequency matches the circular frequency.

(c) We need to solve the initial value problem $\ddot{x} + 6\dot{x} + 9 = 0,\ x(0) = -5,\ \dot{x}(0) = 25$.

$$r^2 + 6r + 9 = (r+3)^2 = 0 \implies r = -3 \text{ with multiplicity } 2$$

$$x(t) = c_1e^{-3t} + c_2te^{-3t} \text{ apply initial conditions}$$

$$x(0) = c_1 + 0 = -5 \implies c_1 = -5$$

$$x(t) = -5e^{-3t} + c_2te^{-3t}$$

$$\dot{x}(t) = 15e^{-3t} - 3c_2te^{-3t} + c_2e^{-3t}$$

$$\dot{x}(0) = 15 + c_2 = 25 \implies c_2 = 10$$

$$x(t) = -5e^{-3t}(1 - 2t)$$

To find when the oscillator passes through the equilibrium position, we solve

$$x(t_0) = 0 = -5e^{-3t_0}(1 - 2t_0) \implies t_0 = \frac{1}{2}$$

The mass passes through the equilibrium position when $t = \frac{1}{2}$ second.

■

7. [2360/050823 (23 pts)] Let $\mathbf{A} = \begin{bmatrix} p & 1 \\ p & p \end{bmatrix}$ where p is a real number.

(a) (5 pts) Is the subset, $\mathbb{W}$, consisting of all matrices of the form $\mathbf{A}$, a subspace of $\mathbb{M}_{22}$? Justify your answer.

(b) (2 pts) For what value(s) of p, if any, is 0 not an eigenvalue of $\mathbf{A}$?

(c) (16 pts) Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.

i. (2 pts) Find Tr $\mathbf{A}$.

ii. (2 pts) Find $|\mathbf{A}|$.

iii. (4 pts) For what value(s) of p will the isolated fixed point (equilibrium solution) at the origin be a saddle?

iv. (4 pts) Classify the geometry and stability of the isolated fixed point at the origin if $p = -1$.

v. (4 pts) For what value(s) of p will there be nonisolated fixed points (equilibrium solutions)?

SOLUTION:

(a) No. $\vec{\mathbf{0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin \mathbb{W}$. It is also not closed under either vector addition or scalar multiplication.

(b) 0 is an eigenvalue of $\mathbf{A}$ if and only if $|\mathbf{A}| = p^2 - p = p(p-1) = 0 \implies p \neq 0, 1$ for 0 to not be an eigenvalue of $\mathbf{A}$.

(c) i. $\text{Tr}\,\mathbf{A} = 2p$

ii. $|\mathbf{A}| = p^2 - p$

iii. To have an isolated fixed point at the origin, we need $|\mathbf{A}| \neq 0 \implies p^2 - p = p(p-1) \neq 0 \implies p \neq 0, 1$. For the isolated fixed point to be a saddle, we need $|\mathbf{A}| < 0 \implies 0 < p < 1$.

iv. If $p = -1$, then $\text{Tr}\,\mathbf{A} = -2$, $|\mathbf{A}| = 2$, $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 2^2 - (4)(2) = -4 < 0$. The fixed point will be an attracting spiral which is stable.

v. We need $|\mathbf{A}| = 0$ which occurs when $p = 0, 1$.

■

8. [2360/050823 (16 pts)] Let $g(t) = \sin \pi t\,[\text{step}(t-2) - \text{step}(t-3)]$.

(a) (4 pts) Write $g(t)$ as a piecewise-defined function.

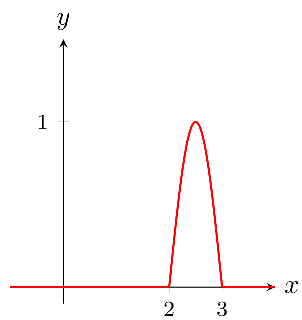
(b) (4 pts) Graph $g(t)$, including appropriate labels.

(c) (8 pts) Find $\mathscr{L}\{g(t)\}$. Note: $\sin(u+v) = \sin u \cos v + \cos u \sin v$

SOLUTION:

(a) $g(t) = \begin{cases} 0 & t < 2 \\ \sin \pi t & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases}$

(b) Graph of $g(t)$



(c)

$$\begin{aligned}
\mathscr{L}\{g(t)\} &= \mathscr{L}\{\sin \pi t\ \text{step}(t-2)\} - \mathscr{L}\{\sin \pi t\ \text{step}(t-3)\} \\
&= e^{-2s}\mathscr{L}\{\sin \pi(t+2)\} - e^{-3s}\mathscr{L}\{\sin \pi(t+3)\} \\
&= e^{-2s}\mathscr{L}\{\sin(\pi t + 2\pi)\} - e^{-3s}\mathscr{L}\{\sin(\pi t + 3\pi)\} \\
&= e^{-2s}\mathscr{L}\{\sin \pi t \cos 2\pi + \cos \pi t \sin 2\pi\} - e^{-3s}\mathscr{L}\{\sin \pi t \cos 3\pi + \cos \pi t \sin 3\pi)\} \\
&= e^{-2s}\mathscr{L}\{\sin \pi t\} - e^{-3s}\mathscr{L}\{-\sin \pi t)\} \\
&= \frac{\pi}{s^2 + \pi^2}\left(e^{-2s} + e^{-3s}\right)
\end{aligned}$$

■

9. [2360/050823 (12 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, writing your answer as a single vector.

Note that the characteristic equation of the coefficient matrix is $(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$.

SOLUTION:

The characteristic equation can be completely factored as $(\lambda - 1)(\lambda - 2)^2 = 0$ implying that the eigenvalues of the coefficient matrix are 1 with multiplicity 1 and 2 with multiplicity 2. We need the eigenvectors.

$$\text{For } \lambda = 1, \text{solve } (\mathbf{A} - \mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}} : \left[\begin{array}{rrr|r} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{For } \lambda = 2, \text{solve } (\mathbf{A} - 2\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}} : \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The general solution is thus

$$\vec{\mathbf{x}}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Applying the initial condition yields

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

with the solution to the initial value problem as

$$\vec{\mathbf{x}}(t) = 4e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4e^t - 3e^{2t} \\ 4e^t - 2e^{2t} \\ 4e^t - e^{2t} \end{bmatrix}$$

■