- This exam is worth 150 points and has 9 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on two sides.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/050823 (18 pts)] Consider the initial value problem $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -2\left(\theta - \theta_s\right),\ \theta(0) = \theta_0$ where θ_s is a constant.
   (a) (8 pts) Solve the initial value problem using the integrating factor method.
   (b) (4 pts) Verify that your answer to part (a) is the solution to the initial value problem.
   (c) (6 pts) Suppose the initial value problem models the rate of change of the temperature, θ, of an object immersed in a water bath with constant temperature θ_s. Suppose the initial temperature of the object, $\theta_0 = 13°\mathrm{C}$, and after $t = \ln 2$ minutes, the object's temperature is $10°\mathrm{C}$. What is the temperature of the water bath, θ_s?

2. [2360/050823 (16 pts)] On a separate page in your bluebook, write the letters (a) through (h) in a column. Then for the following questions, write the word **TRUE** or **FALSE** next to each letter, as appropriate. No partial credit given and no work need be shown. If you do any work to come up with your answers, please do it elsewhere - do not include it in your list of answers (this helps with grading).
   (a) If $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{u}} \in \text{span}\{\vec{\mathbf{v}}, \vec{\mathbf{w}}\}$, then there exist c_1, c_2, c_3, not all zero, such that $c_1\vec{\mathbf{u}} + c_2\vec{\mathbf{v}} + c_3\vec{\mathbf{w}} = \vec{\mathbf{0}}$.
   (b) If $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 4 & 2 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, then $\mathbf{A}^{\mathrm{T}}\left(\mathbf{B} - 2\mathbf{I}\right) = \mathbf{I}$.
   (c) If $|\mathbf{AB}| = 3$, where $\mathbf{A}$ and $\mathbf{B}$ are both square matrices, then $\mathbf{A}$ must be invertible.
   (d) $y = 5$ is a stable equilibrium solution of $y' = \left(y^2 - 4y + 3\right)(y-5)^2$.
   (e) The nullclines of any linear system of differential equations of the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$, where $\mathbf{A}$ is a 2×2 matrix with constant entries (not all zero), are lines through the origin.
   (f) Cramer's Rule can be used to solve a system of three equations in three unknowns with lower triangular coefficient matrix whose diagonal entries are $-1, 0, 2$.
   (g) Picard's Theorem can be used to determine if the initial value problem $y' = \sqrt[3]{t(y-1)},\ y(0) = 1$ has a unique solution.
   (h) $x' = e^{x - \sin t}\cos t$ can be solved using separation of variables.

3. [2360/050823 (20 pts)] The following parts are not related.
   (a) (10 pts) Find the general solution of $x^2 w'' - 2w = 3 - \dfrac{1}{x^2}$ assuming $x > 0$ and solutions to the homogeneous equation have the form $w = x^r$.
   (b) (10 pts) Consider the differential equation $7u'' + 8u' + u = f(t)$. For each $f(t)$ below, give the form only (**do not solve** for the coefficients) of the particular solution guess that would be used in the Method of Undetermined Coefficients. If that method is not possible, write NONE.

      i. $f(t) = t\ln t$ ii. $f(t) = 1$ iii. $f(t) = 5e^{-t/7} + 7e^{-t}$ iv. $f(t) = \cos t - 13\sin 13t$ v. $f(t) = te^{-t}$

4. [2360/050823 (10 pts)] Two 10-liter tanks are initially completely filled with a well-mixed sugar solution. At time $t = 0$, two grams of sugar are dissolved in Tank 1 and one gram of sugar in Tank 2. For $t > 0$, water containing 4 grams of sugar per liter flows into Tank 1 at 3 liters per minute from an outside source. The sweet solution exits the tank system out of Tank 2 at 3 liters per minute. Furthermore, solution from Tank 2 flows into Tank 1 at 4 liters per minute and from Tank 1 into Tank 2 at 7 liters per minute.
   (a) (8 pts) Set up, but **DO NOT** solve, an initial value problem that models the physical situation, writing your final answer using matrices and vectors.
   (b) (2 pts) Without solving the system, over what time interval will the solution be valid?

5. [2360/050823 (10 pts)] Consider the system $\begin{aligned} x_1 + 6x_2 &= 4 \\ 2x_1 + 4x_2 &= -8 \\ -x_1 + 2x_2 &= k \end{aligned}$. Find the value of k that makes the system consistent and find the corresponding solution for that value of k.

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6. [2360/050823 (25 pts)] You have just finished setting up a mass/spring system on a table in your dorm room. The mass is 1 kg and the spring constant is 13 N/m. You have set things up so that the damping constant is 6 N/m/s.

   (a) (15 pts) You have carefully adjusted the oscillator so that the mass itself is at rest at its equilibrium position, you start your stopwatch at $t = 0$ and you are ready to run your experiments. Unfortunately, as you walk away from the table, you bump into it, giving an impulsive force of 10 Newtons to the apparatus when $t = \pi/4$. Where is the mass with respect to its equilibrium position when $t = \pi$?

   (b) (2 pts) After your mishap, you have reset everything as it was originally set up. You now plan on applying a force to the table on which the oscillator lies of $f(t) = \cos\sqrt{13}t$. Once this forcing occurs, will the oscillator exhibit beats, resonance, both or neither? Explain briefly.

   (c) (8 pts) With your experiments done for the day, you go to bed and overnight your roommate sabotages the oscillator, changing out the spring to one whose spring constant is 9 N/m. Upon arising in the morning, you begin your experiments on the unforced oscillator, hoping to see it oscillate nicely. At $t = 0$, you push the mass 5 m to the left of the equilibrium and impart to it a 25 m/s velocity to the right. You know that something is wrong since you see the mass pass through the equilibrium position only once at $t = t_0$. Find the value of t_0.

7. [2360/050823 (23 pts)] Let $\mathbf{A} = \begin{bmatrix} p & 1 \\ p & p \end{bmatrix}$ where p is a real number.

   (a) (5 pts) Is the subset, $\mathbb{W}$, consisting of all matrices of the form $\mathbf{A}$, a subspace of $\mathbb{M}_{22}$? Justify your answer.

   (b) (2 pts) For what value(s) of p, if any, is 0 not an eigenvalue of $\mathbf{A}$?

   (c) (16 pts) Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.

      i. (2 pts) Find Tr $\mathbf{A}$.

      ii. (2 pts) Find $|\mathbf{A}|$.

      iii. (4 pts) For what value(s) of p will the isolated fixed point (equilibrium solution) at the origin be a saddle?

      iv. (4 pts) Classify the geometry and stability of the isolated fixed point at the origin if $p = -1$.

      v. (4 pts) For what value(s) of p will there be nonisolated fixed points (equilibrium solutions)?

8. [2360/050823 (16 pts)] Let $g(t) = \sin \pi t\,[\text{step}(t-2) - \text{step}(t-3)]$.

   (a) (4 pts) Write $g(t)$ as a piecewise-defined function.

   (b) (4 pts) Graph $g(t)$, including appropriate labels.

   (c) (8 pts) Find $\mathscr{L}\{g(t)\}$. Note: $\sin(u+v) = \sin u \cos v + \cos u \sin v$

9. [2360/050823 (12 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, writing your answer as a single vector.

   Note that the characteristic equation of the coefficient matrix is $(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2 + b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2 + b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
