1. [2360/041923 (20 pts)] Consider the initial value problem $y''' - 2y'' = 64e^{-2t},\ y(0) = y'(0) = 0,\ y''(0) = 4$.

   (a) (12 pts) Solve the initial value problem, using the methods of Chapter 4 (that is, do not use Laplace transforms).

   (b) (8 pts) Write the initial value problem as a system of first order differential equations. If possible, write the system, including the initial conditions, in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}},\ \vec{\mathbf{x}}(0) = \vec{\mathbf{x}_0}$. If not possible, say so.

SOLUTION:

(a) The characteristic equation is $r^3 - 2r^2 = r^2(r-2) = 0 \implies r = 0$ (multiplicity 2), $r = 2$ (multiplicity 1). Basis for the solution space of the homogeneous equation is $\{1, t, e^{2t}\}$. Let $y_p = Ae^{-2t}$ and substitute into the nonhomogeneous equation to get

$$-8Ae^{-2t} - 2\left(4Ae^{2t}\right) = -16Ae^{-2t} = 64e^{-2t} \implies A = -4$$

The general solution is $y(t) = c_1 + c_2 t + c_3 e^{2t} - 4e^{-2t}$. Applying the initial conditions yields

$$y(0) = c_1 + c_3 - 4 = 0$$

$$y'(t) = c_2 + 2c_3 e^{2t} + 8e^{-2t} \implies y'(0) = c_2 + 2c_3 + 8 = 0$$

$$y''(t) = 4c_3 e^{2t} - 16e^{-2t} \implies y''(0) = 4c_3 - 16 = 4$$

$$4c_3 = 20 \implies c_3 = 5$$

$$c_2 = -8 - 2(5) = -18$$

$$c_1 = 4 - 5 = -1$$

The solution to the initial value problem is $y(t) = 5e^{2t} - 4e^{-2t} - 18t - 1$.

(b) Let $u_1 = y, u_2 = y', u_3 = y''$. Then

$$\begin{aligned} u_1' &= y' = u_2 \\ u_2' &= y'' = u_3 \\ u_3' &= y''' = 2y'' + 64e^{-2t} = 2u_3 + 64e^{-2t} \end{aligned}$$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 64e^{-2t} \end{bmatrix}, \quad \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

which is in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}},\ \vec{\mathbf{x}}(0) = \vec{\mathbf{x}_0}$.

2. [2360/041923 (24 pts)] On a separate page in your bluebook, write the letters (a) through (l) in a column. Then for the following questions, write the word **TRUE** or **FALSE** next to each letter, as appropriate. No partial credit given and no work need be shown. If you do any work to come up with your answers, please do it elsewhere - do not include it in your list of answers (this helps with grading).

   An harmonic oscillator consisting of a 2-kg mass attached to a spring is horizontally aligned on a table with x measuring the displacement of the mass from its equilibrium position. The damping force is given as $-2p\dot{x}$, where p is a nonnegative real number, and the circular frequency of the oscillator is $\omega_0 = \sqrt{3}$.

   (a) If the oscillator is unforced, the differential equation governing the motion is $2\ddot{x} + 2p\dot{x} + \sqrt{3}x = 0$.

   (b) If $0 \le p < 2\sqrt{3}$, the mass will pass through its equilibrium position more than once if it is given a nonzero initial velocity.

   (c) If $p = 0$ and the oscillator is forced by $f(t) = -3\cos\sqrt{3}t$, then the oscillator will be in resonance.

   (d) If the mass is released from rest 2 meters to the left of its equilibrium position, the initial conditions are $x(0) = 0, \dot{x}(0) = -2$.

   (e) The oscillator will be critically damped only if $p = 2\sqrt{3}$.

   (f) Solutions to the differential equation will be bounded and exhibit beats if $p = 0$ and the oscillator is driven by $f(t) = 100\cos\left[\left(\sqrt{3} - 0.1\right)t\right]$.

   (g) If $p = 0$ and the oscillator is driven by a constant force of F_0, then the system is conservative.

   (h) If the oscillator is driven by $f(t) = F_0 \cos\sqrt{3}t$ (F_0 constant), its solutions will be unbounded for all values of p.

(i) If the oscillator is forced by $f(t) = \dfrac{t}{t+1}$, the particular solution cannot be found using variation of parameters.

(j) If $p = 4$ and the forcing function is $f(t) = e^{-3t} + e^t$, the guess for the particular solution to be used in the method of undetermined coefficients is $y_p = Ate^{-3t} + Be^t$.

(k) If $p > 0$, and the oscillator is unforced, $\lim_{t\to\infty} x(t) = 0$.

(l) For any value of $p \geq 0$, if the oscillator is forced by $f(t) = \cos 20t$, the steady state motion will be oscillatory.

SOLUTION:

(a) **FALSE** $\omega_0 = \sqrt{3} = \sqrt{k/2} \implies k = 6$ so the correct equation is $2\ddot{x} + 2p\dot{x} + 6x = 0$.

(b) **TRUE** The oscillator has to be underdamped or undamped to exhibit oscillatory behavior. To be undamped requires $p = 0$. To be underdamped we need $4p^2 - 4mk < 0 \implies p^2 < mk \implies p < \sqrt{(2)(6)} = 2\sqrt{3}$. Thus $0 \leq p < 2\sqrt{3}$.

(c) **TRUE** The frequency of the forcing function equals the circular frequency of the oscillator.

(d) **FALSE** $x(0) = -2, \dot{x}(0) = 0$

(e) **TRUE** $4p^2 - 4mk = 0 \implies p^2 - (2)(6) = 0 \implies p = 2\sqrt{3}$

(f) **TRUE** The forcing frequency is not equal to the circular frequency, therefore the oscillator is not in resonance and the solutions will be bounded. Since the forcing frequency is "close" to the circular frequency, beats will occur.

(g) **TRUE** In this case, the equation is $2\ddot{x} + 6x = F_0$ or $2\ddot{x} + (6x - F_0) = 0$ which is autonomous and in the form $m\ddot{x} + V'(x) = 0$.

(h) **FALSE** Unbounded solutions (resonance) are only possible in undamped oscillators so p must be zero. (Note: the forcing frequency is correct for resonance).

(i) **FALSE** Variation of parameters must be used since the forcing function is not a member of the family of forcing functions that allow the method of undetermined coefficients to be used.

(j) **TRUE** The differential equation in this case is $2\ddot{x} + 8\dot{x} + 6x = e^{-3t} + e^t$. The characteristic equation is $2r^2 + 8r + 6 = 2(r^2 + 4r + 3) = 2(r+3)(r+1) = 0 \implies r = -3, -1$ giving solutions to the homogeneous equation as e^{-3t}, e^{-t}.

(k) **TRUE** Solutions to all damped, unforced oscillators approach zero as t goes to infinity since the roots of the characteristic equation are either negative real numbers or have negative real parts in case they are complex.

(l) **TRUE** If $p = 0$ the solution will have the form $x(t) = c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t + A\cos 20t + B\sin 20t$, which is clearly oscillatory. If $p > 0$, the solution will look like $x(t) =$ transient solution $+ A\cos 20t + B\sin 20t$, again exhibiting oscillatory behavior in the steady state solution which consists of the last two terms.

3. [2360/041923 (36 pts)] Let $\mathrm{L}(\vec{\mathbf{y}}) = 2y'' - 12y' + 18y$.

(a) (8 pts) Is $\left\{e^{3t}, te^{3t}\right\}$ a basis for the solution space of $\mathrm{L}(\vec{\mathbf{y}}) = 0$? Justify your answer completely.

(b) (12 pts) Use the method of undetermined coefficients to find a particular solution of $\mathrm{L}(\vec{\mathbf{y}}) = 9t^2 - 15$.

(c) (12 pts) Use variation of parameters to find a particular solution of $\mathrm{L}(\vec{\mathbf{y}}) = 12t^{-1}e^{3t}$. Assume $t > 0$.

(d) (4 pts) Find the general solution of $\mathrm{L}(\vec{\mathbf{y}}) = 9t^2 + 12t^{-1}e^{3t} - 15$.

SOLUTION:

(a) Check that both functions are solutions:

$$2\left(e^{3t}\right)'' - 12\left(e^{3t}\right)' + 18e^{3t} = 18e^{3t} - 36e^{3t} + 18e^{3t} = 0$$

$$2\left(te^{3t}\right)'' - 12\left(te^{3t}\right)' + 18te^{3t} = 2(9t+6)e^{3t} - 12(3t+1)e^{3t} + 18te^{3t} = 18te^{3t} + 12e^{3t} - 36te^{3t} - 12e^{3t} + 18te^{3t} = 0$$

or, alternatively, find the roots of the characteristic equation and build the solutions from that:

$$2r^2 - 12r + 8 = 0 \implies 2(r^2 - 6r + 9) = 0 \implies 2(r-3)^2 = 0 \implies r = 3 \text{ with multiplicity } 2$$

giving solutions as e^{3t} and te^{3t}.

Check that the functions are linearly independent:

$$W[e^{3t}, te^{3t}] = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t+1)e^{3t} \end{vmatrix} = e^{6t} \neq 0$$

Since the dimension of the solution space of a second order linear homogeneous differential equation is two and we have two linearly independent solutions, $\left\{e^{3t}, te^{3t}\right\}$ is a basis for the solution space of $\mathrm{L}(\vec{\mathbf{y}}) = 0$.

(b) The form of the particular solution is $y_p = At^2 + Bt + C$. Substituting this into the DE yields

$$2(2A) - 12(2At + B) + 18(At^2 + Bt + C) = 18At^2 + (-24A + 18B)t + 4A - 12B + 18C = 9t^2 - 15$$

$$18A = 9 \implies A = \frac{1}{2}$$

$$-24A + 18B = 0 \implies -24\left(\frac{1}{2}\right) = -18B \implies B = \frac{2}{3}$$

$$4A - 12B + 18C = -15 \implies 4\left(\frac{1}{2}\right) - 12\left(\frac{2}{3}\right) + 15 = -18C \implies C = -\frac{1}{2}$$

Thus $y_{p_1} = \frac{1}{2}t^2 + \frac{2}{3}t - \frac{1}{2}$ is a particular solution.

(c) Before proceeding, we put the differential equation into the form $y'' - 6y' + 9y = 6t^{-1}e^{6t}$ and let $y_1 = e^{3t}, y_2 = te^{3t}$. The right hand side is $f(t) = 6t^{-1}e^{3t}$ and the particular solution will have the form $y_{p_2} = v_1y_1 + v_2y_2$ with

$$v_1 = \int \frac{-te^{3t}(6)t^{-1}e^{3t}}{e^{6t}}\,\mathrm{d}t = -6t$$

$$v_2 = \int \frac{e^{3t}(6)t^{-1}e^{3t}}{e^{6t}}\,\mathrm{d}t = 6\ln|t| = 6\ln t \text{ since } t > 0$$

so that $y_{p_2} = -6te^{3t} + 6te^{3t}\ln t = 6te^{3t}(\ln t - 1)$ is a particular solution.

(d) The general solution is $y = c_1e^{3t} + c_2te^{3t} + 6te^{3t}(\ln t - 1) + \frac{1}{2}t^2 + \frac{2}{3}t - \frac{1}{2} = c_1e^{3t} + \tilde{c}_2te^{3t} + 6te^{3t}\ln t + \frac{1}{2}t^2 + \frac{2}{3}t - \frac{1}{2}$ where $\tilde{c}_2 = c_2 - 6$.

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4. [2360/041923 (20 pts)] Use Laplace transforms to find the solution of $x'' + 2x' + 10x = 10$, $x(0) = 2, x'(0) = -7$. Using any other method of solution will result in zero points. The following may be helpful: $\dfrac{c}{s(s^2 + as + b)} = \dfrac{c}{b}\left(\dfrac{1}{s} - \dfrac{s + a}{s^2 + as + b}\right)$

SOLUTION:

$$\mathscr{L}\{x'' + 2x' + 10x = 10\}$$

$$s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] + 10X(s) = \frac{10}{s}$$

$$\left(s^2 + 2s + 10\right)X(s) - 2s - (-7) - 2(2) = \frac{10}{s}$$

$$\left(s^2 + 2s + 10\right)X(s) = \frac{10}{s} + 2s - 3$$

$$X(s) = \frac{10}{s(s^2 + 2s + 10)} + \frac{2s - 3}{s^2 + 2s + 10}$$

using the given formula for the partial fraction decomposition with $a = 2, b = c = 10$

$$X(s) = \frac{10}{10}\left(\frac{1}{s} - \frac{s + 2}{s^2 + 2s + 10}\right) + \frac{2s - 3}{s^2 + 2s + 10}$$

$$X(s) = \frac{1}{s} + \frac{s - 5}{s^2 + 2s + 10} = \frac{1}{s} + \frac{s + 1 - 6}{(s + 1)^2 + 9}$$

$$X(s) = \frac{1}{s} + \frac{s + 1}{(s + 1)^2 + 3^2} - 2\frac{3}{(s + 1)^2 + 3^2}$$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{s + 1}{(s + 1)^2 + 3^2}\right\} - 2\mathscr{L}^{-1}\left\{\frac{3}{(s + 1)^2 + 3^2}\right\}$$

$$x(t) = 1 + e^{-t}(\cos 3t - 2\sin 3t)$$