- This exam is worth 100 points and has 4 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. $\boxed{\text{NO}}$ calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/041923 (20 pts)] Consider the initial value problem $y''' - 2y'' = 64e^{-2t},\ y(0) = y'(0) = 0,\ y''(0) = 4.$

   (a) (12 pts) Solve the initial value problem, using the methods of Chapter 4 (that is, do not use Laplace transforms).

   (b) (8 pts) Write the initial value problem as a system of first order differential equations. If possible, write the system, including the initial conditions, in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}},\ \vec{\mathbf{x}}(0) = \vec{\mathbf{x}_0}$. If not possible, say so.

2. [2360/041923 (24 pts)] On a separate page in your bluebook, write the letters (a) through (l) in a column. Then for the following questions, write the word **TRUE** or **FALSE** next to each letter, as appropriate. No partial credit given and no work need be shown. If you do any work to come up with your answers, please do it elsewhere - do not include it in your list of answers (this helps with grading).

   An harmonic oscillator consisting of a 2-kg mass attached to a spring is horizontally aligned on a table with x measuring the displacement of the mass from its equilibrium position. The damping force is given as $-2p\dot{x}$, where p is a nonnegative real number, and the circular frequency of the oscillator is $\omega_0 = \sqrt{3}$.

   (a) If the oscillator is unforced, the differential equation governing the motion is $2\ddot{x} + 2p\dot{x} + \sqrt{3}x = 0$.

   (b) If $0 \le p < 2\sqrt{3}$, the mass will pass through its equilibrium position more than once if it is given a nonzero initial velocity.

   (c) If $p = 0$ and the oscillator is forced by $f(t) = -3\cos\sqrt{3}t$, then the oscillator will be in resonance.

   (d) If the mass is released from rest 2 meters to the left of its equilibrium position, the initial conditions are $x(0) = 0, \dot{x}(0) = -2$.

   (e) The oscillator will be critically damped only if $p = 2\sqrt{3}$.

   (f) Solutions to the differential equation will be bounded and exhibit beats if $p = 0$ and the oscillator is driven by $f(t) = 100\cos\left[\left(\sqrt{3} - 0.1\right)t\right]$.

   (g) If $p = 0$ and the oscillator is driven by a constant force of F_0, then the system is conservative.

   (h) If the oscillator is driven by $f(t) = F_0\cos\sqrt{3}t$ (F_0 constant), its solutions will be unbounded for all values of p.

   (i) If the oscillator is forced by $f(t) = \dfrac{t}{t+1}$, the particular solution cannot be found using variation of parameters.

   (j) If $p = 4$ and the forcing function is $f(t) = e^{-3t} + e^{t}$, the guess for the particular solution to be used in the method of undetermined coefficients is $y_p = Ate^{-3t} + Be^{t}$.

   (k) If $p > 0$, and the oscillator is unforced, $\lim\limits_{t\to\infty} x(t) = 0$.

   (l) For any value of $p \ge 0$, if the oscillator is forced by $f(t) = \cos 20t$, the steady state motion will be oscillatory.

3. [2360/041923 (36 pts)] Let $\mathrm{L}(\vec{\mathbf{y}}) = 2y'' - 12y' + 18y$.

   (a) (8 pts) Is $\left\{e^{3t}, te^{3t}\right\}$ a basis for the solution space of $\mathrm{L}(\vec{\mathbf{y}}) = 0$? Justify your answer completely.

   (b) (12 pts) Use the method of undetermined coefficients to find a particular solution of $\mathrm{L}(\vec{\mathbf{y}}) = 9t^2 - 15$.

   (c) (12 pts) Use variation of parameters to find a particular solution of $\mathrm{L}(\vec{\mathbf{y}}) = 12t^{-1}e^{3t}$. Assume $t > 0$.

   (d) (4 pts) Find the general solution of $\mathrm{L}(\vec{\mathbf{y}}) = 9t^2 + 12t^{-1}e^{3t} - 15$.

4. [2360/041923 (20 pts)] Use Laplace transforms to find the solution of $x'' + 2x' + 10x = 10,\ x(0) = 2, x'(0) = -7$. Using any other method of solution will result in zero points. The following may be helpful: $\dfrac{c}{s(s^2+as+b)} = \dfrac{c}{b}\left(\dfrac{1}{s} - \dfrac{s+a}{s^2+as+b}\right)$

LAPLACE TRANSFORM TABLE ON REVERSE

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$