1. [2360/031523 (16 pts)] Some friends of yours are attempting to solve the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. They have already performed a number of elementary row operations on the augmented matrix but are now tired and have asked you complete the problem. Here is the matrix they have given you:

$$\left[\begin{array}{cccc|c} 2 & 2 & -4 & 6 & 10 \\ 0 & 0 & 0 & -4 & 8 \\ -2 & -2 & 4 & -9 & -4 \end{array}\right]$$

   (a) (4 pts) Put the matrix into RREF.

   (b) (4 pts) Find a particular solution to the system.

   (c) (4 pts) Find a basis for the solution space of the associated homogeneous system. What is the dimension of the solution space?

   (d) (4 pts) What is the final solution of the system that you should give your friends? Be sure to write it in the proper form for a nonhomogeneous system.

SOLUTION:

(a)

$$\left[\begin{array}{cccc|c} 2 & 2 & -4 & 6 & 10 \\ 0 & 0 & 0 & -4 & 8 \\ -2 & -2 & 4 & -9 & -4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 2 & -4 & 6 & 10 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 2 & -4 & 0 & 22 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 11 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) $x_2 = s$ and $x_3 = t$ are free variables. We also have $x_1 + x_2 - 2x_3 = 11 \implies x_1 = 11 - s + 2t$ and $x_4 = -2$ as leading/basic variables. A particular solution is $\vec{\mathbf{x}}_p = \begin{bmatrix} 11 & 0 & 0 & -2 \end{bmatrix}^{\mathrm{T}}$

(c) The solution to the associated homogeneous problem is

$$\vec{\mathbf{x}}_h = \begin{bmatrix} -s+2t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

A basis for the solution space of the associated homogeneous problem is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The solution space has dimension 2.

(d)

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 11 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

■

2. [2360/031523 (18 pts)] Let $\mathbf{C} = \begin{bmatrix} 1 & k \\ 2 & 3 \end{bmatrix}$. For each part below, find all real values of k, if any, that make(s) the statement true. No work need be shown and no partial credit available.

   (a) $\mathbf{C}$ is a diagonal matrix

   (b) Tr $\mathbf{C} = 4$

   (c) $\mathbf{C}$ is symmetric

   (d) $\mathbf{C}$ is singular (noninvertible)

   (e) $\mathbf{C} + \begin{bmatrix} 2 & 4 \end{bmatrix}^{\mathrm{T}}$ is defined

   (f) $\mathbf{C}^2 = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix}$

   (g) $\mathbf{C}$ has 2 as an eigenvalue with algebraic multiplicity 2.

(h) $\mathbf{C}$ has $\begin{bmatrix}1\\1\end{bmatrix}$ as an eigenvector with eigenvalue 5.

(i) $\begin{bmatrix}2\\1\end{bmatrix}$ is in the span of the columns of $\mathbf{C}$

SOLUTION:

(a) none

(b) $k \in \mathbb{R}$

(c) $k = 2$

(d) $k = \frac{3}{2}$ (this makes $|\mathbf{C}| = 0$)

(e) none (the matrices are not the same order/size)

(f) $k = 1$ $\quad \mathbf{C}^2 = \begin{bmatrix}1 & k\\2 & 3\end{bmatrix}\begin{bmatrix}1 & k\\2 & 3\end{bmatrix} = \begin{bmatrix}1+2k & 4k\\8 & 2k+9\end{bmatrix}$

(g) $k = -\frac{1}{2}$ $\quad \begin{vmatrix}1-\lambda & k\\2 & 3-\lambda\end{vmatrix} = (1-\lambda)(3-\lambda) - 2k = 0 \implies \lambda^2 - 4\lambda + 3 - 2k = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$

(h) $k = 4$ $\quad \begin{bmatrix}1 & k\\2 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1+k\\5\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix} = 5\begin{bmatrix}1\\1\end{bmatrix}$

(i) $k \neq \frac{3}{2}$ $\quad c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}k\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \to \left[\begin{array}{cc|c}1 & k & 2\\2 & 3 & 1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & k & 2\\0 & 3-2k & -3\end{array}\right]$

■

3. [2360/031523 (19 pts)] Consider the matrix

$$\mathbf{G} = \begin{bmatrix}0 & -1 & 1\\3 & 3c & 12\\1 & 0 & 3\end{bmatrix}$$

(a) (4 pts) Use the cofactor expansion method to show that $|\mathbf{G}| = -3(c+1)$.

(b) (9 pts) Let $\vec{\mathbf{y}} = \begin{bmatrix}a\\3\\c\end{bmatrix}$, $a, c \in \mathbb{R}$ and $\vec{\mathbf{w}} = \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$. Find all values of c and a, if any, such that the system $\mathbf{G}\vec{\mathbf{w}} = \vec{\mathbf{y}}$ has

i. one solution $\qquad$ ii. no solution $\qquad$ iii. an infinite number of solutions

(c) (6 pts) Find w_2 using Cramer's rule when $a = 0,\ c = 0$.

SOLUTION:

(a)

$$|\mathbf{G}| = \begin{vmatrix}0 & -1 & 1\\3 & 3c & 12\\1 & 0 & 3\end{vmatrix} = (-1)(-1)^{1+2}\begin{vmatrix}3 & 12\\1 & 3\end{vmatrix} + (1)(-1)^{1+3}\begin{vmatrix}3 & 3c\\1 & 0\end{vmatrix}$$

$$= (1)(-3) + 1(-3c) = -3 - 3c = -3(c+1)$$

(b) i. To have only one solution, $|\mathbf{G}| \neq 0$ meaning $c \neq -1, a \in \mathbb{R}$.

ii. For this case, $c = -1$ and we will need the RREF.

$$\left[\begin{array}{ccc|c}0 & -1 & 1 & a\\3 & -3 & 12 & 3\\1 & 0 & 3 & -1\end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{R_2^* = -3R_3 + R_2} \left[\begin{array}{ccc|c}1 & 0 & 3 & -1\\0 & -3 & 3 & 6\\0 & -1 & 1 & a\end{array}\right] \xrightarrow[\substack{R_3^* = -R_3\\R_3 \leftrightarrow R_2}]{R_2^* = -3R_3 + R_2} \left[\begin{array}{ccc|c}1 & 0 & 3 & -1\\0 & 1 & -1 & -a\\0 & 0 & 0 & -3(a-2)\end{array}\right]$$

The system will have no solutions if $a \neq 2$.

iii. Using the information from part (ii), $c = -1$ and $a = 2$ for a consistent system with infinitely many solutions.

(c)

$$w_2 = \frac{\begin{vmatrix}0 & 0 & 1\\3 & 3 & 12\\1 & 0 & 3\end{vmatrix}}{\begin{vmatrix}0 & -1 & 1\\3 & 0 & 12\\1 & 0 & 3\end{vmatrix}} = \frac{1(-1)^{1+3}\begin{vmatrix}3 & 3\\1 & 0\end{vmatrix}}{-1(-1)^{1+2}\begin{vmatrix}3 & 12\\1 & 3\end{vmatrix}} = \frac{-3}{-3} = 1$$

4. [2360/031523 (10 pts)] Determine, with appropriate justification, if the following subsets, $\mathbb{W}$, are subspaces of the given vector space, $\mathbb{V}$.

(a) (5 pts) $\mathbb{V} = \mathbb{R}^2$; $\mathbb{W} = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, (x-1)^2 + y^2 \leq 1 \right\}$ [the set of points in the disk of radius 1 centered at $(1, 0)$]

(b) (5 pts) $\mathbb{V} = \mathbb{M}_{22}$; $\mathbb{W}$ the set 2×2 matrices of the form $\begin{bmatrix} a & a \\ 2a & 2a \end{bmatrix}$ where $a \in \mathbb{R}$

SOLUTION:

(a) No, $\mathbb{W}$ is not a subspace of $\mathbb{V}$. The set is not closed under scalar multiplication nor vector addition. For example, the point $(1, 0)$ is in the set but $4(1, 0) = (4, 0)$ is not. Or $(1, 0), (2, 0)$ are in the set but $(1, 0) + (2, 0) = (3, 0)$ is not.

(b) Yes, $\mathbb{W}$ is a subspace of $\mathbb{V}$. The set is closed under both scalar multiplication and vector addition (equivalently under linear combinations). Let $p, q \in \mathbb{R}$ and $\vec{\mathbf{u}} = \begin{bmatrix} a & a \\ 2a & 2a \end{bmatrix}, \vec{\mathbf{v}} = \begin{bmatrix} b & b \\ 2b & 2b \end{bmatrix} \in \mathbb{W}$. Then

$$p\vec{\mathbf{u}} + q\vec{\mathbf{v}} = \begin{bmatrix} pa & pa \\ 2pa & 2pa \end{bmatrix} + \begin{bmatrix} qb & qb \\ 2qb & 2qb \end{bmatrix} = \begin{bmatrix} pa + qb & pa + qb \\ 2(pa + qb) & 2(pa + qb) \end{bmatrix} \in \mathbb{W}$$

5. [2360/031523 (17 pts)] The follows parts (a) and (b) are not related. All problems require justification.

(a) (5 pts) Is the set of vectors $\{\sin t, \sin 2t\}$ linearly independent on the real line?

(b) (12 pts) Let $\vec{\mathbf{p}}_1 = t^2 + 2$, $\vec{\mathbf{p}}_2 = -3t^2 + 2t$, $\vec{\mathbf{p}}_3 = -t - 3$ and $\vec{\mathbf{0}} = 0t^2 + 0t + 0$.

i. (4 pts) Show that $c_1\vec{\mathbf{p}}_1 + c_2\vec{\mathbf{p}}_2 + c_3\vec{\mathbf{p}}_3 = \vec{\mathbf{0}}$ has the solution $c_1 = 6, c_2 = 2, c_3 = 4$.

ii. (4 pts) Using the result from part (i), show that $\vec{\mathbf{p}}_2 \in \text{span}\{\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_3\}$.

iii. (4 pts) Answer YES or NO to the following question and provide a brief written justification for your answer: The set $\{\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3\}$ forms a basis for $\mathbb{P}_2$.

SOLUTION:

(a) Use the Wronskian.

$$W[\sin t, \sin 2t] = \begin{vmatrix} \sin t & \sin 2t \\ \cos t & 2\cos 2t \end{vmatrix} = 2\sin t \cos 2t - \cos t \sin 2t$$

Evaluating this at $t = \frac{\pi}{2}$ yields -2 showing that the Wronskian is nonzero for at least one value of $t \in \mathbb{R}$, implying that the vectors are linearly independent on the real line.

(b) i.

$$\begin{aligned} 6\vec{\mathbf{p}}_1 + 2\vec{\mathbf{p}}_2 + 4\vec{\mathbf{p}}_3 &= 6\left(t^2 + 2\right) + 2\left(-3t^2 + 2t\right) + 4(-t - 3) \\ &= 6t^2 + 12 - 6t^2 + 4t - 4t - 12 \\ &= 0t^2 + 0t + 0 \\ &= \vec{\mathbf{0}} \end{aligned}$$

Alternatively, we can solve $c_1\left(t^2 + 2\right) + c_2\left(-3t^2 + 2t\right) + c_3(-t - 3) = 0t^2 + 0t + 0$. Equating coefficients of like terms on both sides yields

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 2 & 0 & -3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} c_1 = 3c_2 \\ c_2 = \frac{1}{2}c_3 \\ c_3 = t \end{array}$$

one solution of which is $c_1 = 6, c_2 = 2, c_3 = 4$.

ii.

$$\begin{aligned} 6\vec{\mathbf{p}}_1 + 2\vec{\mathbf{p}}_2 + 4\vec{\mathbf{p}}_3 &= \vec{\mathbf{0}} \\ 2\vec{\mathbf{p}}_2 &= -6\vec{\mathbf{p}}_1 - 4\vec{\mathbf{p}}_3 \\ \vec{\mathbf{p}}_2 &= -3\vec{\mathbf{p}}_1 - 2\vec{\mathbf{p}}_3 \end{aligned}$$

iii. NO. The dimension of $\mathbb{P}_2$ is 3 so a basis must contain 3 linearly independent vectors. The set here, although containing 3 vectors, is linearly dependent and thus cannot constitute a basis for $\mathbb{P}_2$.

■

6. [2360/031523 (20 pts)] Consider two invertible matrices $\mathbf{A}$ and $\mathbf{B}$ whose inverses are given as

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Hint: No credit for using elementary row operations in parts (b) and (c).

(a) (8 pts) For $\mathbf{B}^{-1}$, find the eigenspace associated with the eigenvalue possessing an algebraic multiplicity of two.

(b) (4 pts) Compute $|\mathbf{AB}|$

(c) (8 pts) Solve $\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

SOLUTION:

(a) Since the matrix is upper triangular, its eigenvalues lie on the diagonal. The eigenvalue with algebraic multiplicity of 2 is 3. We find nontrivial solutions of $\left(\mathbf{B}^{-1} - 3\mathbf{I}\right)\vec{\mathbf{v}} = \vec{\mathbf{0}}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \mathbb{E}_{\lambda=3} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

(b)

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = \left(\frac{1}{|\mathbf{A}^{-1}|}\right)\left(\frac{1}{|\mathbf{B}^{-1}|}\right) = \left(\frac{1}{-5}\right)\left(\frac{1}{9}\right) = -\frac{1}{45}$$

(c)

$$\vec{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \mathbf{B}^{-1}\left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\right)^{\mathrm{T}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 5 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 11 \\ 5 & 0 & 5 \\ 9 & -3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$$

■