- This exam is worth 100 points and has 6 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/031523 (16 pts)] Some friends of yours are attempting to solve the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. They have already performed a number of elementary row operations on the augmented matrix but are now tired and have asked you complete the problem. Here is the matrix they have given you:

$$\left[\begin{array}{cccc|c} 2 & 2 & -4 & 6 & 10 \\ 0 & 0 & 0 & -4 & 8 \\ -2 & -2 & 4 & -9 & -4 \end{array}\right]$$

   (a) (4 pts) Put the matrix into RREF.

   (b) (4 pts) Find a particular solution to the system.

   (c) (4 pts) Find a basis for the solution space of the associated homogeneous system. What is the dimension of the solution space?

   (d) (4 pts) What is the final solution of the system that you should give your friends? Be sure to write it in the proper form for a nonhomogeneous system.

2. [2360/031523 (18 pts)] Let $\mathbf{C} = \begin{bmatrix} 1 & k \\ 2 & 3 \end{bmatrix}$. For each part below, find all real values of k, if any, that make(s) the statement true. No work need be shown and no partial credit available.

   (a) $\mathbf{C}$ is a diagonal matrix

   (b) Tr $\mathbf{C} = 4$

   (c) $\mathbf{C}$ is symmetric

   (d) $\mathbf{C}$ is singular (noninvertible)

   (e) $\mathbf{C} + \begin{bmatrix} 2 & 4 \end{bmatrix}^{\mathrm{T}}$ is defined

   (f) $\mathbf{C}^2 = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix}$

   (g) $\mathbf{C}$ has 2 as an eigenvalue with algebraic multiplicity 2.

   (h) $\mathbf{C}$ has $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an eigenvector with eigenvalue 5.

   (i) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in the span of the columns of $\mathbf{C}$

3. [2360/031523 (19 pts)] Consider the matrix

$$\mathbf{G} = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 3c & 12 \\ 1 & 0 & 3 \end{bmatrix}$$

   (a) (4 pts) Use the cofactor expansion method to show that $|\mathbf{G}| = -3(c+1)$.

   (b) (9 pts) Let $\vec{\mathbf{y}} = \begin{bmatrix} a \\ 3 \\ c \end{bmatrix}$, $a, c \in \mathbb{R}$ and $\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Find all values of c and a, if any, such that the system $\mathbf{G}\vec{\mathbf{w}} = \vec{\mathbf{y}}$ has

      i. one solution   ii. no solution   iii. an infinite number of solutions

   (c) (6 pts) Find w_2 using Cramer's rule when $a = 0,\ c = 0$.

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4. [2360/031523 (10 pts)] Determine, with appropriate justification, if the following subsets, $\mathbb{W}$, are subspaces of the given vector space, $\mathbb{V}$.

   (a) (5 pts) $\mathbb{V} = \mathbb{R}^2$; $\mathbb{W} = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, (x-1)^2 + y^2 \leq 1 \right\}$ [the set of points in the disk of radius 1 centered at $(1, 0)$]

   (b) (5 pts) $\mathbb{V} = \mathbb{M}_{22}$; $\mathbb{W}$ the set 2×2 matrices of the form $\begin{bmatrix} a & a \\ 2a & 2a \end{bmatrix}$ where $a \in \mathbb{R}$

5. [2360/031523 (17 pts)] The follows parts (a) and (b) are not related. All problems require justification.

   (a) (5 pts) Is the set of vectors $\{\sin t, \sin 2t\}$ linearly independent on the real line?

   (b) (12 pts) Let $\vec{\mathbf{p}}_1 = t^2 + 2$, $\vec{\mathbf{p}}_2 = -3t^2 + 2t$, $\vec{\mathbf{p}}_3 = -t - 3$ and $\vec{\mathbf{0}} = 0t^2 + 0t + 0$.

      i. (4 pts) Show that $c_1\vec{\mathbf{p}}_1 + c_2\vec{\mathbf{p}}_2 + c_3\vec{\mathbf{p}}_3 = \vec{\mathbf{0}}$ has the solution $c_1 = 6, c_2 = 2, c_3 = 4$.

      ii. (4 pts) Using the result from part (i), show that $\vec{\mathbf{p}}_2 \in \text{span}\{\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_3\}$.

      iii. (4 pts) Answer YES or NO to the following question and provide a brief written justification for your answer: The set $\{\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3\}$ forms a basis for $\mathbb{P}_2$.

6. [2360/031523 (20 pts)] Consider two invertible matrices $\mathbf{A}$ and $\mathbf{B}$ whose inverses are given as

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Hint: No credit for using elementary row operations in parts (b) and (c).

   (a) (8 pts) For $\mathbf{B}^{-1}$, find the eigenspace associated with the eigenvalue possessing an algebraic multiplicity of two.

   (b) (4 pts) Compute $|\mathbf{AB}|$

   (c) (8 pts) Solve $\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$