1. [2360/021523 (12 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The operator $\mathrm{L}(\vec{\mathbf{y}}) = y'' + e^{t^2} y' + (\cos t)\, y$ satisfies the two properties of linear operators.

   (b) A population of bacteria growing exponentially doubles in population every 10 days. Therefore, the population size is described by the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 10y$, where $t = 1$ corresponds to 1 day.

   (c) Let $y' = f(y)$, where f is continuous. An $f(y)$ exists such that there are only two equilibrium solutions, both of which are stable.

   (d) The equation $y' + 2y + 1 = 0$ is a linear, first order, nonhomogeneous, autonomous differential equation.

   (e) The differential equation $t^{-3}\,(y')^2 + e^t y = \cos(t)$ can be solved via the integrating factor method.

   (f) The isoclines of the differential equation $y' = t^2 + 1$ are parabolas and all solutions, regardless of the initial condition, are unbounded (approach infinity) as t goes to infinity.

SOLUTION:

(a) **TRUE**

$$
\begin{aligned}
\mathrm{L}\left(k\vec{\mathbf{y}}\right) &= (ky)'' + e^{t^2}\,(ky)' + (\cos t)\,(ky) \\
&= ky'' + ke^{t^2}y' + k\,(\cos t)\,y \\
&= k\left[y'' + e^{t^2}y' + (\cos t)\,y\right] \\
&= k\mathrm{L}\left(\vec{\mathbf{y}}\right) \\
\mathrm{L}\left(\vec{\mathbf{x}} + \vec{\mathbf{y}}\right) &= (x+y)'' + e^{t^2}\,(x+y)' + (\cos t)\,(x+y) \\
&= x'' + y'' + e^{t^2}x' + e^{t^2}y' + (\cos t)\,x + (\cos t)\,y \\
&= \left[x'' + e^{t^2}x' + (\cos t)\,x\right] + \left[y'' + e^{t^2}y' + (\cos t)\,y\right] \\
&= \mathrm{L}\left(\vec{\mathbf{x}}\right) + \mathrm{L}\left(\vec{\mathbf{y}}\right)
\end{aligned}
$$

(b) **FALSE** The correct equation describing this situation is $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \left(\dfrac{\ln 2}{10}\right) y$.

(c) **FALSE** If there were two equilibrium solutions, both of which were stable, then y' would have to take on both negative and positive values between them, an impossibility.

(d) **TRUE** Rewritten as $y' + 2y = -1$ shows that it is nonhomogeneous and rewriting as $y' = -2y - 1$ shows that it is in the form $y' = f(y)$.

(e) **FALSE** The equation is nonlinear due to the presence of the $(y')^2$ term and therefore the integrating factor method cannot be used to solve it.

(f) **FALSE** The isoclines are $t^2 + 1 = c$, which are the vertical lines $t = \pm\sqrt{c-1}$. Note that $c \geq 1$. Because of this, $y' > 0$ everywhere, implying that solutions are increasing functions for all t, or, equivalently, all solutions are unbounded, regardless of the initial condition.

■

2. [2360/021523 (16 pts)] The following questions are unrelated.

   (a) (8 pts) Compute the equilibrium solutions of the differential equation $y' = (1 - y^2)y$ and classify their stability.

   (b) (8 pts) Given the initial value problem

$$
y' = (y - t)^{2/3}, \quad y(t_0) = y_0,
$$

   for which initial conditions (t_0, y_0) are we guaranteed that there exists a unique solution to the initial value problem?

SOLUTION:

(a) Rewrite the equation as $y' = (1+y)(1-y)y$ giving the equilibrium solutions as the roots of $(1+y)(1-y)y = 0$ which are $y = -1, 0, 1$. We then have

$$y' \begin{cases} > 0, & y < -1 \\ < 0, & -1 < y < 0 \\ > 0, & 0 < y < 1 \\ < 0, & y > 1 \end{cases} \implies y = 0 \text{ is unstable and } y = \pm 1 \text{ are stable}$$

(b) We have $f(t, y) = (y-t)^{2/3}$ which is continuous everywhere and thus for any initial data (t_0, y_0). Picard's theorem therefore guarantees the existence of a solution for any initial value problem. However, $f_y = \frac{2}{3}(y-t)^{-1/3}$ is not defined and therefore not continuous if $y = t$. Thus for any initial data with $t_0 = y_0$ we cannot guarantee that a unique solution exists. For any initial data such that $t_0 \neq y_0$ Picard's theorem does guarantee a unique solution to the initial value problem.

■

3. [2360/021523 (15 pts)] Consider the differential equation $y' - \sqrt{y} = -e^{-t}\sqrt{y}$.

(a) (3 pts) Find all equilibrium solutions, if any exist.

(b) (12 pts) Find the general solution of the equation, writing your answer in explicit form, that is, $y(t) = \cdots$.

SOLUTION:

(a) Rewriting the DE as $y' = \sqrt{y}\,(1 - e^{-t})$ indicates that $y = 0$ is the only equilibrium solution.

(b) Use separation of variables, assuming $y \neq 0$.

$$\begin{aligned} y' &= \sqrt{y} - e^{-t}\sqrt{y} = \sqrt{y}\,(1 - e^{-t}) \\ \int y^{-1/2}\,\mathrm{d}y &= \int (1 - e^{-t})\,\mathrm{d}t \\ 2y^{1/2} &= t + e^{-t} + C \\ y(t) &= \frac{1}{4}\left(t + e^{-t} + C\right)^2 \end{aligned}$$

■

4. [2360/021523 (15 pts)] Let $\cos x \dfrac{\mathrm{d}y}{\mathrm{d}x} + (\sin x)y = 1$, with $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Use the Euler-Lagrange two stage method (variation of parameters) to find the solution of the equation that passes through the point $\left(\dfrac{\pi}{3}, 0\right)$.

SOLUTION:

Solve the associated homogeneous equation.

$$\begin{aligned} \cos x \frac{\mathrm{d}y_h}{\mathrm{d}x} + (\sin x)y_h &= 0 \\ \int \frac{\mathrm{d}y_h}{y_h} &= -\int \frac{\sin x}{\cos x}\,\mathrm{d}x \quad (u = \cos x, \mathrm{d}u = -\sin x\,\mathrm{d}x) \\ \ln|y_h| &= \ln|\cos x| + K \\ \ln\left|\frac{y_h}{\cos x}\right| &= K \\ \left|\frac{y_h}{\cos x}\right| &= e^K \\ y_h &= C\cos x \end{aligned}$$

Now set $y_p = v(x)\cos x$ and substitute into the nonhomogeneous equation.

$$\begin{aligned} (\cos x)y_p' + (\sin x)y_p &= -v\sin x\cos x + v'\cos^2 x + v\cos x\sin x = 1 \\ v' &= \frac{1}{\cos^2 x} = \sec^2 x \\ v(x) &= \tan x \end{aligned}$$

so we have $y_p = \tan x \cos x = \sin x$ and using the Nonhomogeneous Principle, $y = y_h + y_p = C\cos x + \sin x$. Applying the initial condition $y\left(\frac{\pi}{3}\right) = 0$ yields

$$0 = C\cos\frac{\pi}{3} + \sin\frac{\pi}{3} = C\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2} \implies C = -\sqrt{3}$$

giving the solution to the problem as $y = -\sqrt{3}\cos x + \sin x$. ■

5. [2360/021523 (12 pts)] A swimming pool with a capacity of 5000 gallons (gal) initially $(t = 0)$ contains 1000 gal of fresh water. Water containing te^{-t} pounds (lb) of chlorine per gallon is entering the pool at a rate of 4 gal/min, and the well mixed chlorinated water is allowed to drain out the bottom of the pool at a rate of 2 gal/min. Let $x(t)$ denote the amount (lb) of chlorine in the pool at any time t.

   (a) (10 pts) Write down, but **do not solve**, the initial value problem that $x(t)$ satisfies.

   (b) (2 pts) Without solving the initial value problem from part (a), state the interval over which the solution will be valid based on the physical situation.

SOLUTION:

(a) Since the inflow and outflow differ, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow in} - \text{flow out} = 4 - 2 = 2, \quad V(0) = 1000$$

$$\int \mathrm{d}V = \int 2\,\mathrm{d}t$$

$$V(t) = 2t + C$$

$$V(0) = 0 + C = 1000 \implies C = 1000$$

$$V(t) = 1000 + 2t$$

Then we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{mass rate in} - \text{mass rate out}$$

$$= te^{-t}(4) - \frac{x}{1000 + 2t}(2)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{500 + t} = 4te^{-t}, \quad x(0) = 0$$

where the initial condition comes from the fact that there is no chlorine in the pool at time $t = 0$.

(b) The solution will be valid until the pool fills, or $5000 = 2t + 1000 \implies t = 2000$ minutes. Solution is valid on the interval $0 \le t \le 2000$ or $t \in [0, 2000]$.

■

6. [2360/021523 (14 pts)] Consider the system of differential equations

$$x' = 2x + xy$$

$$y' = -y + x^2 y$$

   (a) (3 pts) Find the h nullclines.

   (b) (3 pts) Find the v nullclines.

   (c) (5 pts) On a single phase plane, plot the h nullclines as solid curves/lines and the v nullclines as dashed curves/lines. Label all intercepts.

   (d) (3 pts) Find the equilibrium points, if any exist.

SOLUTION:

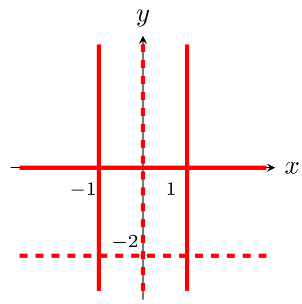
(a) h nullclines are where $y' = 0$.

$$-y + x^2y = y(-1 + x^2) = y(x+1)(x-1) = 0 \implies y = 0, x = \pm 1$$

(b) v nullclines are where $x' = 0$.

$$2x + xy = x(2 + y) = 0 \implies x = 0, y = -2$$

(c) Solid lines are h nullclines and dashed lines are v nullclines.



(d) Equilibrium points are where the h and v nullclines intersect, or equivalently, where $x' = y' = 0$ simultaneously. These occur at $(0, 0), (1, -2), (-1, -2)$.

■

7. [2360/021523 (16 pts)] The following problems are unrelated.

(a) (8 pts) Find the general solution of the differential equation $(1 + x)z' + z = 1 - x^2,\ x > -1$ using the integrating factor method.

(b) (8 pts) If $x' + e^{\sin 2t}x = 1$ and $x(\pi/4) = 2$, approximate $x(\pi/2)$ using one step of Euler's method.

SOLUTION:

(a)

$$z' + \frac{1}{1+x}z = 1 - x$$

$$\int \frac{\mathrm{d}x}{1+x} = \ln|1+x| \implies \mu(x) = e^{\ln|1+x|} = 1 + x \quad \text{since } x > -1$$

$$\int [(1+x)z]' = \int \left(1 - x^2\right) \mathrm{d}x$$

$$(1+x)z = x - \frac{x^3}{3} + C$$

$$z(x) = \frac{x - x^3/3 + C}{x + 1}$$

(b) Writing the equation as $x' = 1 - e^{\sin 2t}x$ we have $f(t, x) = 1 - e^{\sin 2t}x$. From the given information we have $h = \pi/4$ so that Euler's method yields

$$x_{n+1} = x_n + \frac{\pi}{4}\left(1 - e^{\sin 2t_n}x_n\right)$$

$$x(\pi/2) \approx x_1 = 2 + \frac{\pi}{4}\left[1 - e^{\sin(2\pi/4)}(2)\right] = 2 + \frac{\pi}{4}(1 - 2e)$$

■