- This exam is worth 100 points and has 7 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5”$\times$ 11” crib sheet with writing on one side.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: ”I will abide by the CU Boulder Honor Code on this exam.” **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/021523 (12 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The operator $\mathrm{L}(\vec{\mathbf{y}}) = y'' + e^{t^2}y' + (\cos t)\, y$ satisfies the two properties of linear operators.

   (b) A population of bacteria growing exponentially doubles in population every 10 days. Therefore, the population size is described by the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 10y$, where $t = 1$ corresponds to 1 day.

   (c) Let $y' = f(y)$, where f is continuous. An $f(y)$ exists such that there are only two equilibrium solutions, both of which are stable.

   (d) The equation $y' + 2y + 1 = 0$ is a linear, first order, nonhomogeneous, autonomous differential equation.

   (e) The differential equation $t^{-3}\left(y'\right)^2 + e^t y = \cos(t)$ can be solved via the integrating factor method.

   (f) The isoclines of the differential equation $y' = t^2 + 1$ are parabolas and all solutions, regardless of the initial condition, are unbounded (approach infinity) as t goes to infinity.

2. [2360/021523 (16 pts)] The following questions are unrelated.

   (a) (8 pts) Compute the equilibrium solutions of the differential equation $y' = (1 - y^2)y$ and classify their stability.

   (b) (8 pts) Given the initial value problem
   $$y' = (y - t)^{2/3}, \quad y(t_0) = y_0,$$
   for which initial conditions (t_0, y_0) are we guaranteed that there exists a unique solution to the initial value problem?

3. [2360/021523 (15 pts)] Consider the differential equation $y' - \sqrt{y} = -e^{-t}\sqrt{y}$.

   (a) (3 pts) Find all equilibrium solutions, if any exist.

   (b) (12 pts) Find the general solution of the equation, writing your answer in explicit form, that is, $y(t) = \cdots$.

4. [2360/021523 (15 pts)] Let $\cos x \dfrac{\mathrm{d}y}{\mathrm{d}x} + (\sin x)y = 1$, with $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Use the Euler-Lagrange two stage method (variation of parameters) to find the solution of the equation that passes through the point $\left(\dfrac{\pi}{3}, 0\right)$.

5. [2360/021523 (12 pts)] A swimming pool with a capacity of 5000 gallons (gal) initially $(t = 0)$ contains 1000 gal of fresh water. Water containing te^{-t} pounds (lb) of chlorine per gallon is entering the pool at a rate of 4 gal/min, and the well mixed chlorinated water is allowed to drain out the bottom of the pool at a rate of 2 gal/min. Let $x(t)$ denote the amount (lb) of chlorine in the pool at any time t.

   (a) (10 pts) Write down, but **do not solve**, the initial value problem that $x(t)$ satisfies.

   (b) (2 pts) Without solving the initial value problem from part (a), state the interval over which the solution will be valid based on the physical situation.

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6. [2360/021523 (14 pts)] Consider the system of differential equations

$$x' = 2x + xy$$
$$y' = -y + x^2 y$$

(a) (3 pts) Find the h nullclines.

(b) (3 pts) Find the v nullclines.

(c) (5 pts) On a single phase plane, plot the h nullclines as solid curves/lines and the v nullclines as dashed curves/lines. Label all intercepts.

(d) (3 pts) Find the equilibrium points, if any exist.

7. [2360/021523 (16 pts)] The following problems are unrelated.

(a) (8 pts) Find the general solution of the differential equation $(1+x)z' + z = 1 - x^2,\ x > -1$ using the integrating factor method.

(b) (8 pts) If $x' + e^{\sin 2t}x = 1$ and $x(\pi/4) = 2$, approximate $x(\pi/2)$ using one step of Euler's method.