1. [2360/043022 (20 pts)] Solve the following initial value problem: $y'' + 3t^2 = 5\delta(t-2) + 6(t-4)\text{step}\,(t-4)$, $y(0) = 2$, $y'(0) = 0$.

SOLUTION:

$$\mathscr{L}\left\{y'' + 3t^2 = 5\delta(t-2) + 6(t-4)\text{step}\,(t-4)\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + \frac{6}{s^3} = 5e^{-2s} + \frac{6e^{-4s}}{s^2}$$

$$Y(s) = \frac{5e^{-2s}}{s^2} + \frac{6e^{-4s}}{s^4} - \frac{6}{s^5} + \frac{2}{s}$$

$$y(t) = \mathscr{L}^{-1}\left\{5e^{-2s}\frac{1!}{s^{1+1}} + e^{-4s}\frac{3!}{s^{3+1}} - \frac{6}{4!}\frac{4!}{s^{4+1}} + \frac{2}{s}\right\}$$

$$y(t) = 5(t-2)\text{step}(t-2) + (t-4)^3\text{step}\,(t-4) - \frac{1}{4}t^4 + 2$$

■

2. [2360/043022 (15 pts)] Solve the system $\begin{cases} x + 2z = 1 \\ y - 4z = 1 \\ x + z = 2 \end{cases}$ by using Gauss-Jordan elimination to find the inverse of an appropriate matrix. No credit awarded if any other technique is used (*e.g.*, RREF, Cramer's Rule).

SOLUTION:

The system can be written in the form $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ as

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

so that $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 4 & 1 & -4 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & -4 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix}$$

■

3. [2360/043022 (18 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) For invertible matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, if $\mathbf{AB} = \mathbf{CA}$, then $|\mathbf{B}| = |\mathbf{C}|$.

   (b) The space of all solutions to the differential equation $y'' - \left(\cos t^2\right) y' + y - 5t = 0$ forms a vector space (usual addition and scalar multiplication are assumed).

   (c) $\mathbf{A}$ is a 3×4 matrix. When the system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent, then we must have infinitely many solutions. Here $\vec{\mathbf{x}}$ is a 4×1 column vector and $\vec{\mathbf{b}}$ is a 3×1 column vector.

   (d) The equilibrium solution $x = 1$ of the differential equation $x'(t) = x(1-x)$ is unstable.

   (e) If $f(t, y)$ is continuous everywhere, Picard's theorem guarantees that the differential equation $y' = f(t, y)$ has a unique solution for any initial condition $y(t_0) = y_0$.

   (f) If 1 is an eigenvalue of a 3×3 matrix $\mathbf{A}$, then $\mathbf{A}$ must be invertible.

   (g) The function $f(t) = \begin{cases} 1 & t < 2 \\ e^t & 2 \le t < 3 \\ 2 & t \ge 3 \end{cases}$ can be written as $f(t) = 1 + (e^t - 1)\text{step}(t-2) + (2 - e^t)\text{step}(t-3)$.

   (h) The set of all 2×2 diagonal matrices is a dimension 4 subspace of $\mathbb{M}_{22}$.

   (i) If the functions $\{y_1(t), y_2(t), y_3(t)\}$ are all solutions to a third order linear homogeneous differential equation and $W[y_1, y_2, y_3](t) \equiv 0$, then the functions cannot form a basis for the solution solution space.

SOLUTION:

(a) TRUE Since all of the matrices are invertible, their respective determinants are nonzero. Thus

$$\mathbf{AB} = \mathbf{CA} \implies |\mathbf{AB}| = |\mathbf{CA}| \implies |\mathbf{A}||\mathbf{B}| = |\mathbf{C}||\mathbf{A}| \implies \cancel{|\mathbf{A}|}|\mathbf{B}| = |\mathbf{C}|\cancel{|\mathbf{A}|} \implies |\mathbf{B}| = |\mathbf{C}|$$

(b) FALSE The differential equation is nonhomogeneous.

(c) TRUE The system is underdetermined. Underdetermined systems never have unique solutions.

(d) FALSE If $x > 1$, $x' < 0$. If $x < 1$, $x' > 0$. The equilibrium solution is stable.

(e) FALSE Uniqueness is guaranteed only if $f_y(t, y)$ is continuous in a rectangle surrounding (t_0, y_0).

(f) FALSE 0 could be an eigenvalue in which case the matrix would be singular.

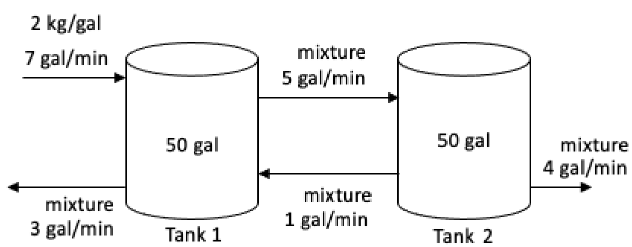
(g) TRUE

(h) FALSE The set is a subspace but its dimension is 2.

(i) TRUE The functions are linearly dependent.

■

4. [2360/043022 (12 pts)] Two fifty gallon tanks are full of Kool-Aid® solution. Initially, Tank 1 contains 5 kg of dissolved Kool-Aid® powder and Tank 2 has 3 kg of dissolved Kool-Aid® powder in it. The well-stirred solution is pumped between the tanks as shown in the figure below. Construct, but **DO NOT SOLVE**, a mathematical model for the number of kilograms of Kool-Aid® powder $x_1(t)$ and $x_2(t)$ at time t in Tanks 1 and 2, respectively. Write your final answer using matrices and vectors.



SOLUTION:

The initial conditions are $x_1(t) = 5$ and $x_2(t) = 3$. Since the net flow into and out of each tank is zero, the volume of solution in the tanks remains constant at 50 gal. Using the relationship rate of change = rate in − rate out we have

$$x_1'(t) = \left(2\frac{\text{kg}}{\text{gal}}\right)\left(7\frac{\text{gal}}{\text{min}}\right) + \left(\frac{x_2}{50}\frac{\text{kg}}{\text{gal}}\right)\left(1\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x_1}{50}\frac{\text{kg}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x_1}{50}\frac{\text{kg}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right)$$

$$x_2'(t) = \left(\frac{x_1}{50}\frac{\text{kg}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x_2}{50}\frac{\text{kg}}{\text{gal}}\right)\left(1\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x_2}{50}\frac{\text{kg}}{\text{gal}}\right)\left(4\frac{\text{gal}}{\text{min}}\right)$$

or

$$x_1'(t) = -\frac{4}{25}x_1(t) + \frac{1}{50}x_2(t) + 14$$

$$x_2'(t) = \frac{1}{10}x_1(t) - \frac{1}{10}x_2(t)$$

Using matrices and vectors,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{4}{25} & \frac{1}{50} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

■

5. [2360/043022 (15 pts)] Use the integrating factor method to solve the initial value problem $xy' + 3y = -4x^{-4}$, $y(1) = -4$ on the interval $x > 0$.

SOLUTION:

$$xy' + 3y = -4x^{-4}$$

$$y' + \frac{3}{x}y = -4x^{-5} \qquad \text{integrating factor } \mu(x) = e^{\int \frac{3}{x}\,\mathrm{d}x} = x^3$$

$$\left(x^3 y\right)' = -4x^{-2}$$

$$x^3 y = -4\int x^{-2}\,\mathrm{d}x = 4x^{-1} + C \qquad \text{apply initial condition}$$

$$(1)^3(-4) = 4(1)^{-1} + C \implies C = -8$$

$$y(x) = 4x^{-4} - 8x^{-3} = \frac{4}{x^4}(1 - 2x)$$

■

6. [2360/043022 (20 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$. Write your final answer as a single vector.

SOLUTION:

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = \lambda^2(2-\lambda) - 4(2-\lambda) = (2-\lambda)(\lambda^2 - 4) = (2-\lambda)(\lambda-2)(\lambda+2) = 0 \implies$$

so the eigenvalues are 2 with multiplicity 2 and -2 with multiplicity 1.

$$\lambda = -2: \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 + v_3 = 0 \\ v_2 = 0 \\ v_3 = r \in \mathbb{R} \end{array} \implies \vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 2: \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 - v_3 = 0 \\ v_2 = r \in \mathbb{R} \\ v_3 = s \in \mathbb{R} \end{array} \implies \vec{\mathbf{v}} = r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \implies \vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The general solution is thus

$$\vec{\mathbf{x}}(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Applying the initial condition gives

$$c_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$$

so that

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Thus,

$$\vec{\mathbf{x}}(t) = 2e^{-2t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 2e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2e^{-2t} + 4e^{2t} \\ 2e^{2t} \\ 2e^{-2t} + 4e^{2t} \end{bmatrix}$$

■

7. [2360/043022 (12 pts)] Convert $t^2 y''' - (\cos t)y' = 2$, $y(0) = 3$, $y'(0) = -1$, $y''(0) = 4$ with $t > 0$ into a system of first order differential equations. Write your final answer using matrices, if possible.

SOLUTION:

Let $u_1 = y, u_2 = y', u_3 = y''$. Then

$$u_1' = y' = u_2$$

$$u_2' = y'' = u_3$$

$$u_3' = y''' = \frac{2}{t^2} + \frac{\cos t}{t^2}y' = \frac{\cos t}{t^2}u_2 + \frac{2}{t^2}$$

Since the original equation is linear, we can write this system/initial value problem using matrices as

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{\cos t}{t^2} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{2}{t^2} \end{bmatrix}, \ \vec{\mathbf{u}}(0) = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

■

8. [2360/043022 (20 pts)] Consider the linear system of differential equations given by $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} a-1 & 1 \\ a-2 & 1 \end{bmatrix}$ (a is a real number) and with equilibrium solution $\vec{\mathbf{x}}_* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   (a) Is $\vec{\mathbf{x}}_*$ the only possible equilibrium solution? Justify your answer.

   (b) For what value(s) of a, if any, will the equilibrium solution $\vec{\mathbf{x}}_*$ be a saddle?

   (c) For what value(s) of a, if any, will the equilibrium solution $\vec{\mathbf{x}}_*$ be unstable?

   (d) For what value(s) of a, if any, will the equilibrium solution $\vec{\mathbf{x}}_*$ be an attracting node?

SOLUTION:

Note that $|\mathbf{A}| = a - 1 - (a-2) = 1$ and $\text{Tr}\mathbf{A} = a - 1 + 1 = a$

   (a) Yes, since $|\mathbf{A}| \neq 0$, the only equilibrium solution is $\vec{\mathbf{x}}_*$.

   (b) For a saddle we need $|\mathbf{A}| < 0$. No values of a result in a saddle.

   (c) To be unstable, since $|\mathbf{A}| > 0$ we need $\text{Tr}\mathbf{A} > 0 \implies a > 0$.

   (d) To be an attracting node, we need $\text{Tr}\mathbf{A} < 0$ and $(\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| > 0 \implies a^2 - 4 > 0 \implies |a| > 2 \implies a > 2$ or $a < -2$. For an attracting node, we need $\text{Tr}\mathbf{A} < 0$ so $a < -2$.

■

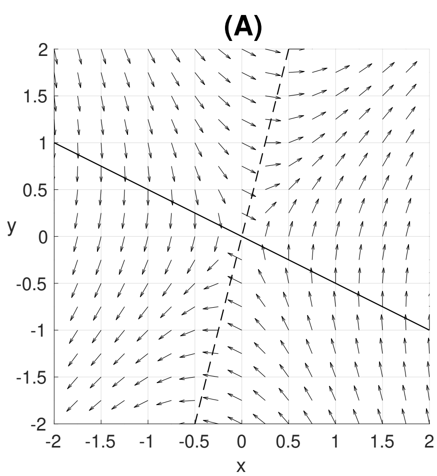
9. [2360/043022 (18 pts)] The following parts of this problem are related.

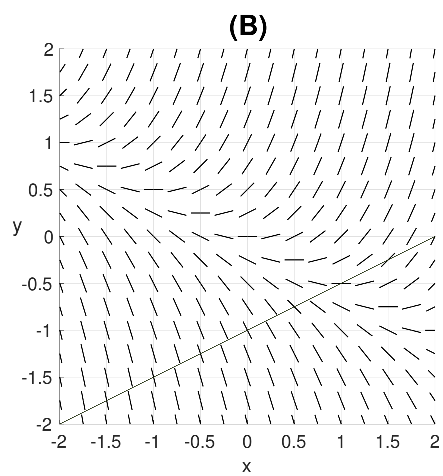
   (a) In your bluebook, write the letters A-D. Next to each letter, write the Roman numeral corresponding to the equation or system depicted in each figure. Each letter corresponds to exactly one Roman numeral.

   I. $\begin{array}{l} x' = x - 2y \\ y' = 4x - y \end{array}$ II. $\begin{array}{l} x' = x + 2y \\ y' = -4x + y \end{array}$ III. $y' = x + 2y$ IV. $\begin{array}{l} x' = x + 2y \\ y' = 4x - y \end{array}$ V. $y' = x - 2y$ VI. $y' = 2x - y$

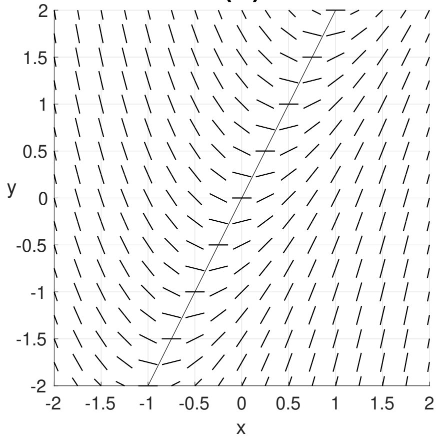
   (b) What do the lines in graph A represent? Be sure to differentiate between solid and dashed in your answer.

   (c) What does the line in graph C represent?

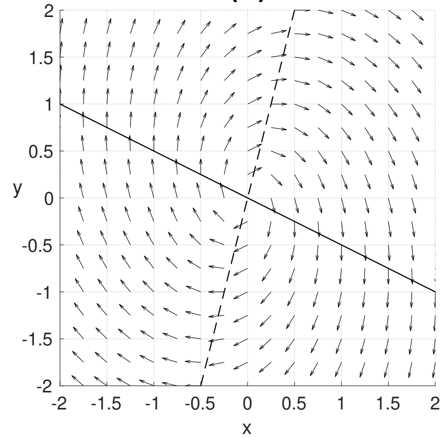




(C)



(D)



SOLUTION:

(a) (A) IV (B) III (C) VI (D) II

(b) The solid line is the v nullcline and the dashed line is the h nullcline.

(c) The line is the isocline depicting where the slope of the solution curve is zero (the solution curve is horizontal)