- This exam is worth 150 points and has 9 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on both sides.

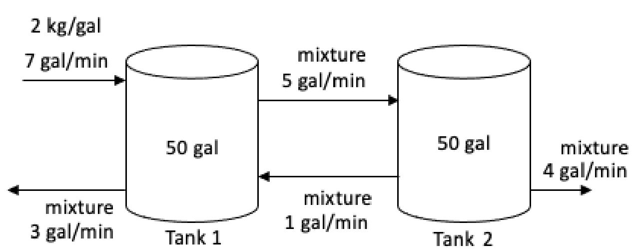
At the top of the page containing your answer to problem 1, write the following italicized statement and sign your name to it: "*I will abide by the CU Boulder Honor Code on this exam.*" **FAILURE TO INCLUDE THIS MAY RESULT IN A PENALTY.**

1. [2360/043022 (20 pts)] Solve the following initial value problem: $y'' + 3t^2 = 5\delta(t-2) + 6(t-4)\text{step}\,(t-4),\ y(0) = 2,\ y'(0) = 0.$

2. [2360/043022 (15 pts)] Solve the system $\begin{cases} x + 2z = 1 \\ y - 4z = 1 \\ x + z = 2 \end{cases}$ by using Gauss-Jordan elimination to find the inverse of an appropriate matrix. No credit awarded if any other technique is used (*e.g.*, RREF, Cramer's Rule).

3. [2360/043022 (18 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.
   (a) For invertible matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, if $\mathbf{AB} = \mathbf{CA}$, then $|\mathbf{B}| = |\mathbf{C}|$.
   (b) The space of all solutions to the differential equation $y'' - \left(\cos t^2\right) y' + y - 5t = 0$ forms a vector space (usual addition and scalar multiplication are assumed).
   (c) $\mathbf{A}$ is a 3×4 matrix. When the system of linear equations $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent, then we must have infinitely many solutions. Here $\vec{\mathbf{x}}$ is a 4×1 column vector and $\vec{\mathbf{b}}$ is a 3×1 column vector.
   (d) The equilibrium solution $x = 1$ of the differential equation $x'(t) = x(1-x)$ is unstable.
   (e) If $f(t, y)$ is continuous everywhere, Picard's theorem guarantees that the differential equation $y' = f(t, y)$ has a unique solution for any initial condition $y(t_0) = y_0$.
   (f) If 1 is an eigenvalue of a 3×3 matrix $\mathbf{A}$, then $\mathbf{A}$ must be invertible.
   (g) The function $f(t) = \begin{cases} 1 & t < 2 \\ e^t & 2 \le t < 3 \\ 2 & t \ge 3 \end{cases}$ can be written as $f(t) = 1 + (e^t - 1)\text{step}(t-2) + (2 - e^t)\text{step}(t-3)$.
   (h) The set of all 2×2 diagonal matrices is a dimension 4 subspace of $\mathbb{M}_{22}$.
   (i) If the functions $\{y_1(t), y_2(t), y_3(t)\}$ are all solutions to a third order linear homogeneous differential equation and $W[y_1, y_2, y_3](t) \equiv 0$, then the functions cannot form a basis for the solution solution space.

4. [2360/043022 (12 pts)] Two fifty gallon tanks are full of Kool-Aid® solution. Initially, Tank 1 contains 5 kg of dissolved Kool-Aid® powder and Tank 2 has 3 kg of dissolved Kool-Aid® powder in it. The well-stirred solution is pumped between the tanks as shown in the figure below. Construct, but **DO NOT SOLVE**, a mathematical model for the number of kilograms of Kool-Aid® powder $x_1(t)$ and $x_2(t)$ at time t in Tanks 1 and 2, respectively. Write your final answer using matrices and vectors.



5. [2360/043022 (15 pts)] Use the integrating factor method to solve the initial value problem $xy' + 3y = -4x^{-4}$, $y(1) = -4$ on the interval $x > 0$.

6. [2360/043022 (20 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$. Write your final answer as a single vector.

7. [2360/043022 (12 pts)] Convert $t^2 y''' - (\cos t)y' = 2,\ y(0) = 3,\ y'(0) = -1,\ y''(0) = 4$ with $t > 0$ into a system of first order differential equations. Write your final answer using matrices, if possible.

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8. [2360/043022 (20 pts)] Consider the linear system of differential equations given by $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} a-1 & 1 \\ a-2 & 1 \end{bmatrix}$ (a is a real number) and with equilibrium solution $\vec{\mathbf{x}}_* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(a) Is $\vec{\mathbf{x}}_*$ the only possible equilibrium solution? Justify your answer.

(b) For what value(s) of a, if any, will the equilibrium solution $\vec{\mathbf{x}}_*$ be a saddle?

(c) For what value(s) of a, if any, will the equilibrium solution $\vec{\mathbf{x}}_*$ be unstable?

(d) For what value(s) of a, if any, will the equilibrium solution $\vec{\mathbf{x}}_*$ be an attracting node?

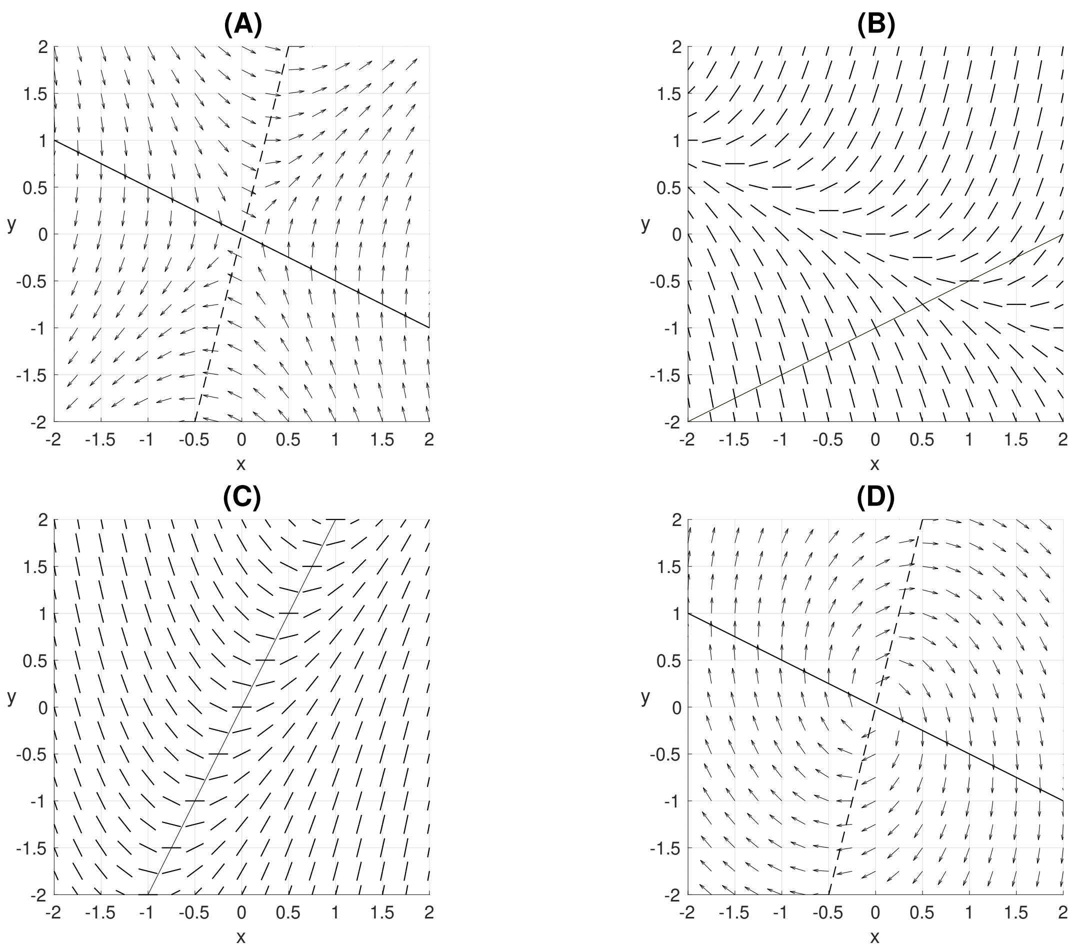
9. [2360/043022 (18 pts)] The following parts of this problem are related.

(a) In your bluebook, write the letters A-D. Next to each letter, write the Roman numeral corresponding to the equation or system depicted in each figure. Each letter corresponds to exactly one Roman numeral.

I. $\begin{array}{l} x' = x - 2y \\ y' = 4x - y \end{array}$ II. $\begin{array}{l} x' = x + 2y \\ y' = -4x + y \end{array}$ III. $y' = x + 2y$ IV. $\begin{array}{l} x' = x + 2y \\ y' = 4x - y \end{array}$ V. $y' = x - 2y$ VI. $y' = 2x - y$

(b) What do the lines in graph A represent? Be sure to differentiate between solid and dashed in your answer.

(c) What does the line in graph C represent?



Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$