1. [2360/041322 (20 pts)] Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 8t,\ y(0) = 3$ using Laplace transforms. No points will be awarded if any other technique is used.

SOLUTION:

$$\mathscr{L}\{y'\} - 2\mathscr{L}\{y\} = -8\mathscr{L}\{t\}$$
$$sY(s) - y(0) - 2Y(s) = -\frac{8}{s^2}$$
$$(s-2)Y(s) = -\frac{8}{s^2} + 3$$
$$Y(s) = -\frac{8}{s^2(s-2)} + \frac{3}{s-2}$$

Partial fraction decomposition on the first term

$$\frac{8}{s^2(s-2)} = \frac{As+B}{s^2} + \frac{C}{s-2}$$
$$8 = (As+B)(s-2) + Cs^2$$
$$s = 0:\ 8 = -2B \implies B = -4$$
$$s = 2:\ 8 = 4C \implies C = 2$$
$$s = 1:\ 8 = (A-4)(-1) + 2 \implies A = -2$$
$$\frac{8}{s^2(s-2)} = -\frac{2}{s} - \frac{4}{s^2} + \frac{2}{s-2}$$

We then have

$$Y(s) = \frac{2}{s} + \frac{4}{s^2} + \frac{1}{s-2}$$
$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + 4\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$$
$$y(t) = 2 + 4t + e^{2t}$$

■

2. [2360/041322 (34 pts)] Consider the initial value problem $x^2y'' + 3xy' - 3y = x^{-1},\ y(1) = \frac{11}{4},\ y'(1) = -\frac{3}{4}$ with $x > 0$.

   (a) (8 pts) Assuming solutions of the form $y = x^r$, find a basis for the solution space of the associated homogeneous problem.

   (b) (6 pts) Verify that your answer to part (a) is indeed a basis.

   (c) (12 pts) Find the general solution to the original nonhomogeneous equation.

   (d) (8 pts) Solve the initial value problem.

SOLUTION:

(a) With $y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$. Substituting these into the associated homogeneous equation yields

$$x^2\left[r(r-1)x^{r-2}\right] + 3xrx^{r-1} - 3x^r = x^r\left(r^2 + 2r - 3\right) = 0 \implies (r+3)(r-1) = 0 \implies r = -3, 1$$

A basis is thus $\left\{x^{-3}, x\right\}$.

(b) We are considering a second order linear homogeneous differential equation so the dimension of the solution space is 2. Check that the two functions are indeed solutions:

$$x^2\left(x^{-3}\right)'' + 3x\left(x^{-3}\right)' - 3x^{-3} = x^2\left(12x^{-5}\right) + 3x\left(-3x^{-4}\right) - 3x^{-3} = 0$$
$$x^2(x)'' + 3x(x)' - 3x = x^2(0) + 3x - 3x = 0$$

Check that the functions are linearly independent

$$W\left[x^{-3}, x\right] = \begin{vmatrix} x^{-3} & x \\ -3x^{-4} & 1 \end{vmatrix} = 4x^{-3} \neq 0$$

so the two functions are linearly independent. $\left\{x^{-3}, x\right\}$ is a basis for the solution space.

(c) We need to use variation of parameters so start by putting the differential equation into standard form

$$y'' + \frac{3}{x}y' - \frac{3}{x^2}y = \frac{1}{x^3}$$

We'll let $y_1 = x^{-3}$ and $y_2 = x$, $f(x) = x^{-3}$ from the differential equation and $W\left[y_1, y_2\right] = 4x^{-3}$ from part (b). The particular solution will have the form $y_p = v_1 y_1 + v_2 y_2$ where

$$v_1' = \frac{-y_2 f}{W\left[y_1, y_2\right]} = \frac{-x\left(x^{-3}\right)}{4x^{-3}} = -\frac{x}{4} \implies v_1 = \int -\frac{x}{4}\,\mathrm{d}x = -\frac{x^2}{8}$$

$$v_2' = \frac{y_1 f}{W\left[y_1, y_2\right]} = \frac{x^{-3}\left(x^{-3}\right)}{4x^{-3}} = \frac{x^{-3}}{4} \implies v_2 = \int \frac{x^{-3}}{4}\,\mathrm{d}x = -\frac{x^{-2}}{8}$$

giving $y_p = \left(-\frac{x^2}{8}\right)x^{-3} - \left(\frac{x^{-2}}{8}\right)x = -\frac{1}{4x}$ and general solution $y = y_h + y_p = c_1 x^{-3} + c_2 x - \frac{1}{4x}$

(d)

$$y(1) = c_1 + c_2 - \frac{1}{4} = \frac{11}{4} \implies c_1 + c_2 = 3$$

$$y'(1) = -3c_1 + c_2 + \frac{1}{4} = -\frac{3}{4} \implies -3c_1 + c_2 = -1$$

yields $c_1 = 1, c_2 = 2$ so the solution to the IVP is $y = x^{-3} + 2x - \frac{1}{4x} = \frac{8x^4 - x^2 + 4}{4x^3}$. ■

3. [2360/041322 (20 pts)] Use the Method of Undetermined Coefficients to find the general solution of $2y'' - 2y' + 5y = 10t - 27e^{2t}$.

SOLUTION:

Solve the associated homogeneous problem first. The characteristic equation is

$$2r^2 - 2r + 5 = 0 \implies r = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(5)}}{4} = \frac{2 \pm \sqrt{-36}}{4} = \frac{1}{2} \pm \frac{3}{2}i$$

$$y_h = e^{t/2}\left(c_1 \cos\frac{3}{2}t + c_2 \sin\frac{3}{2}t\right)$$

The particular solution has the form $y_p = At + B + Ce^{2t}$. Substituting this into the equation yields

$$2y_p'' - 2y_p' + 5y_p = 8Ce^{2t} - 2\left(A + 2Ce^{2t}\right) + 5\left(At + B + Ce^{2t}\right) = 9Ce^{2t} + 5At + 5B - 2A = 10t - 27e^{2t}$$

$$\implies A = 2, B = \frac{5}{4}, C = -3 \implies y_p = 2t + \frac{4}{5} - 3e^{2t}$$

The general solution is thus $y = y_h + y_p = e^{t/2}\left(c_1 \cos\frac{3}{2}t + c_2 \sin\frac{3}{2}t\right) - 3e^{2t} + 2t + \frac{4}{5}$. ■

4. [2360/041322 (14 pts)] A certain harmonic oscillator consists of an object attached to a wall via a spring. The entire apparatus is arranged horizontally on a table. The displacement, $x(t)$, of the mass is governed by the differential equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0\cos(\alpha t)$, where ω_0 is the circular frequency.

(a) (4 pts) Find the initial conditions if the object is released from rest 1 unit to the left of its equilibrium position.

(b) (4 pts) Find all values of β, ω_0, F_0 and α that will put the oscillator in resonance.

(c) (3 pts) Suppose $F_0 > 0$, $\alpha < 0$ and $\beta > 0$. No justification required and no partial credit given.

i. Will solutions of the differential equation remain bounded?

ii. How many times will the object pass through the equilibrium position?

iii. Will solutions contain both a transient and steady state solution?

(d) (3 pts) If $\beta = 3$ and the mass of the object is 4, what is the value of the spring/restoring constant if the oscillator is critically damped?

SOLUTION:

(a) $x(0) = -1,\ \dot{x}(0) = 0$

(b) $\beta = 0,\ \omega_0 > 0,\ F_0 \neq 0,\ \alpha = \omega_0$

(c) i. yes

ii. infinitely many

iii. yes

(d) Critically damped means $4\beta^2 - 4\omega_0^2 = 0 \implies 9 - \frac{k}{4} = 0 \implies k = 36$

■

5. [2360/041322 (12 pts)] Characteristic equations for certain constant coefficient linear homogeneous differential equations are given along with a forcing function, $f(t)$. Give the form of the particular solution you would use to solve the nonhomogeneous [with the given $f(t)$] differential equation from which the characteristic equation was derived using the Method of Undetermined Coefficients. Do not solve for the coefficients.

(a) (4 pts) $r(3r-1) = 0;\ f(t) = -3 + \sin t$

(b) (4 pts) $r^2 + 2r + 5 = [r - (-1+2i)][r - (-1-2i)]) = 0;\ f(t) = e^{-t} + 5\cos 2t$

(c) (4 pts) $(r-3)^3(r+2) = 0;\ f(t) = te^{3t} + 2e^{-2t}$

SOLUTION:

(a) Homogeneous solutions are $1, e^{t/3};\ y_p = At + B\cos t + C\sin t$

(b) Homogeneous solutions are $e^{-t}\cos 2t, e^{-t}\sin 2t;\ y_p = Ae^{-t} + B\cos 2t + C\sin 2t$

(c) Homogeneous solutions are $e^{3t}, te^{3t}, t^2e^{3t}, e^{-2t};\ y_p = t^3(At+B)e^{3t} + Cte^{-2t}$

■