- This exam is worth 100 points and has 5 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on one side.

At the top of the page containing your answer to problem 1, write the following italicized statement and sign your name to it: "*I will abide by the CU Boulder Honor Code on this exam.*" **FAILURE TO INCLUDE THIS MAY RESULT IN A PENALTY.**

1. [2360/041322 (20 pts)] Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 8t$, $y(0) = 3$ using Laplace transforms. No points will be awarded if any other technique is used.

2. [2360/041322 (34 pts)] Consider the initial value problem $x^2y'' + 3xy' - 3y = x^{-1}$, $y(1) = \frac{11}{4}$, $y'(1) = -\frac{3}{4}$ with $x > 0$.
   (a) (8 pts) Assuming solutions of the form $y = x^r$, find a basis for the solution space of the associated homogeneous problem.
   (b) (6 pts) Verify that your answer to part (a) is indeed a basis.
   (c) (12 pts) Find the general solution to the original nonhomogeneous equation.
   (d) (8 pts) Solve the initial value problem.

3. [2360/041322 (20 pts)] Use the Method of Undetermined Coefficients to find the general solution of $2y'' - 2y' + 5y = 10t - 27e^{2t}$.

4. [2360/041322 (14 pts)] A certain harmonic oscillator consists of an object attached to a wall via a spring. The entire apparatus is arranged horizontally on a table. The displacement, $x(t)$, of the mass is governed by the differential equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0\cos(\alpha t)$, where ω_0 is the circular frequency.
   (a) (4 pts) Find the initial conditions if the object is released from rest 1 unit to the left of its equilibrium position.
   (b) (4 pts) Find all values of β, ω_0, F_0 and α that will put the oscillator in resonance.
   (c) (3 pts) Suppose $F_0 > 0$, $\alpha < 0$ and $\beta > 0$. No justification required and no partial credit given.
      i. Will solutions of the differential equation remain bounded?
      ii. How many times will the object pass through the equilibrium position?
      iii. Will solutions contain both a transient and steady state solution?
   (d) (3 pts) If $\beta = 3$ and the mass of the object is 4, what is the value of the spring/restoring constant if the oscillator is critically damped?

5. [2360/041322 (12 pts)] Characteristic equations for certain constant coefficient linear homogeneous differential equations are given along with a forcing function, $f(t)$. Give the form of the particular solution you would use to solve the nonhomogeneous [with the given $f(t)$] differential equation from which the characteristic equation was derived using the Method of Undetermined Coefficients. Do not solve for the coefficients.
   (a) (4 pts) $r(3r - 1) = 0;\ f(t) = -3 + \sin t$
   (b) (4 pts) $r^2 + 2r + 5 = [r - (-1 + 2i)][r - (-1 - 2i)]) = 0;\ f(t) = e^{-t} + 5\cos 2t$
   (c) (4 pts) $(r - 3)^3(r + 2) = 0;\ f(t) = te^{3t} + 2e^{-2t}$

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\left\{\delta(t-c)\right\} = e^{-cs}$$

$$\mathscr{L}\left\{t f'(t)\right\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\left\{f(t-c)\,\mathrm{step}(t-c)\right\} = e^{-cs}F(s) \qquad \mathscr{L}\left\{f(t)\,\mathrm{step}(t-c)\right\} = e^{-cs}\mathscr{L}\left\{f(t+c)\right\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$