1. [2360/030922 (10 pts)] Given the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ -1 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -1 & 4 \end{bmatrix}$$

write the word **TRUE** or **FALSE** as appropriate. No work need be shown, no work will be graded and no partial credit will be given.

(a) $\mathbf{CB} = \begin{bmatrix} -2 \\ 5 \\ 11 \end{bmatrix}$ (b) $\text{Tr}\left(\mathbf{B}^{\text{T}}\mathbf{A}^{\text{T}}\right) = 2$ (c) $\mathbf{A}^{\text{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\text{T}}$ (d) $|\mathbf{C}^{\text{T}}\mathbf{C} - 3\mathbf{I}| = -10$ (e) $\mathbf{AB} - \mathbf{A}^{\text{T}}\mathbf{B}^{\text{T}}$ is not defined

SOLUTION:

(a) FALSE $\mathbf{CB} = \begin{bmatrix} -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 11 \end{bmatrix}$

(b) TRUE $\text{Tr}\left(\mathbf{B}^{\text{T}}\mathbf{A}^{\text{T}}\right) = \text{Tr}\left(\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -2 \end{bmatrix}\right) = \text{Tr}\begin{bmatrix} 2 & 6 & -2 \\ -1 & 1 & -1 \\ -3 & -1 & -1 \end{bmatrix} = 2 + 1 - 1 = 2$

(c) FALSE $\mathbf{A}^{\text{T}}\mathbf{A}$ is $(2 \times 3)(3 \times 2) = 2 \times 2$ whereas $\mathbf{A}\mathbf{A}^{\text{T}}$ is $(3 \times 2)(2 \times 3) = 3 \times 3$ so they cannot be equal

(d) FALSE $|\mathbf{C}^{\text{T}}\mathbf{C} - 3\mathbf{I}| = \left|\begin{bmatrix} -1 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 4 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} 1 & -4 \\ -4 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right| = \begin{vmatrix} -2 & -4 \\ -4 & 13 \end{vmatrix} = -42$

(e) TRUE $\mathbf{AB}$ is $(3 \times 2)(2 \times 3) = 3 \times 3$ whilst $\mathbf{A}^{\text{T}}\mathbf{B}^{\text{T}}$ is $(2 \times 3)(3 \times 2) = 2 \times 2$ so the subtraction is not defined

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2. [2360/030922 (12 pts)] Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

(a) (4 pts) Find the eigenvalues of $\mathbf{A}$ and state the multiplicity (also known as the algebraic multiplicity) of each.

(b) (8 pts) Find the dimension of and a basis for the eigenspace associated with the eigenvalue whose (algebraic) multiplicity is greater than 1.

SOLUTION:

(a)

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 0 & 3 \\ 0 & 3-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = (3-\lambda)(-1)^{2+2}\begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 9) = -(\lambda - 3)^2(\lambda + 3) = 0$$

Eigenvalues are $\lambda = -3$ with algebraic multiplicity 1 and $\lambda = 3$ with algebraic multiplicity 2.

(b) We need to solve the system $(\mathbf{A} - 3\mathbf{I})\vec{\mathbf{v}} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} s \\ t \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad s, t \in \mathbb{R}$$

A basis for the eigenspace is $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ with a dimension of 2.

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3. [2360/030922 (14 pts)] Let $\vec{\mathbf{p}}_1 = 1 + x^2$, $\vec{\mathbf{p}}_2 = x - x^2$, $\vec{\mathbf{p}}_3 = 2 + 2x + 4x^2$. Show that $\vec{\mathbf{p}} = 3 + 4x - 2x^2$ is in span $\{\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3\}$ by writing $\vec{\mathbf{p}}$ as a linear combination of $\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3$. Use Cramer's Rule and cofactor expansion to solve an appropriate linear system.

SOLUTION:

We need to find constants c_1, c_2, c_3 such that $c_1\vec{\mathbf{p}}_1 + c_2\vec{\mathbf{p}}_2 + c_3\vec{\mathbf{p}}_3 = \vec{\mathbf{p}}$ or

$$c_1(1+x^2) + c_2(x-x^2) + c_3(2+2x+4x^2) = 3+4x-2x^2$$

Equating coefficients on each side yields the linear system

$$\begin{aligned} 1c_1 + 0c_2 + 2c_3 &= 3 \\ 0c_1 + 1c_2 + 2c_3 &= 4 \\ 1c_1 - 1c_2 + 4c_3 &= -2 \end{aligned}$$

or written using matrices as $\mathbf{A}\vec{\mathbf{c}} = \vec{\mathbf{b}}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

Cramer's Rule gives:

$$c_1 = \frac{\begin{vmatrix} 3 & 0 & 2 \\ 4 & 1 & 2 \\ -2 & -1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & -1 & 4 \end{vmatrix}} = \frac{3(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} 4 & 1 \\ -2 & -1 \end{vmatrix}}{1(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} + 1(-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix}} = \frac{14}{4} = \frac{7}{2}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 3 & 2 \\ 0 & 4 & 2 \\ 1 & -2 & 4 \end{vmatrix}}{4} = \frac{1(-1)^{1+1}\begin{vmatrix} 4 & 2 \\ -2 & 4 \end{vmatrix} + 1(-1)^{3+1}\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix}}{4} = \frac{18}{4} = \frac{9}{2}$$

$$c_3 = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 1 & -1 & -2 \end{vmatrix}}{4} = \frac{1(-1)^{1+1}\begin{vmatrix} 1 & 4 \\ -1 & -2 \end{vmatrix} + 1(-1)^{3+1}\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix}}{4} = -\frac{1}{4}$$

Thus, $\vec{\mathbf{p}} = \frac{7}{2}\vec{\mathbf{p}}_1 + \frac{9}{2}\vec{\mathbf{p}}_2 - \frac{1}{4}\vec{\mathbf{p}}_3$.

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4. [2360/030922 (14 pts)] Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. NO credit will be given if Gauss-Jordan elimination is used.

(a) (5 pts) Using only matrix multiplication, verify that $\mathbf{B} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix}$ is the inverse of $\mathbf{A}$.

(b) (9 pts) Using only matrix multiplication and properties of the matrix inverse and transpose, solve $\mathbf{A}^{\mathrm{T}}\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

SOLUTION:

(a)

$$\mathbf{AB} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \quad \text{or} \quad \mathbf{BA} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

(b)

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{A}\vec{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \text{Note: } \left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\mathbf{A}^{\mathrm{T}} = \mathbf{I} \text{ and } \mathbf{IA} = \mathbf{A}$$

$$\mathbf{A}\vec{\mathbf{x}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{x}} = \mathbf{A}^{-1}\left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \text{Note: } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \text{ and } \mathbf{I}\vec{\mathbf{x}} = \vec{\mathbf{x}}$$

$$\vec{\mathbf{x}} = \mathbf{A}^{-1}\left(\mathbf{A}^{-1}\right)^{\mathrm{T}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \text{Note: } \left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}$$

$$\vec{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\vec{\mathbf{x}} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 1 & 0 \\ -5 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -18 \end{bmatrix}$$

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5. [2360/030922 (12 pts)] Determine if each of the following sets of vectors forms a basis for $\mathbb{R}^3$. Justify your answers.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ -2 \end{bmatrix} \right\}$

SOLUTION:

Note that the dimension of $\mathbb{R}^3$ is 3 so a basis consists of 3 linearly independent vectors.

(a) The set contains only 2 vectors and thus cannot form a basis for $\mathbb{R}^3$ regardless of the linear dependence or independence of the vectors in the set.

(b) Three vectors in $\mathbb{R}^3$ can potentially be a basis if they are linearly independent. To check for this, we need to see if the only solution to

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 3 & -3 \\ 2 & -1 & 8 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is the trivial solution. The determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 3 & -3 \\ 2 & -1 & 8 \\ 0 & 1 & -2 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} -1 & 8 \\ 1 & -2 \end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix} 3 & -3 \\ 1 & -2 \end{vmatrix} = 1(1)(-6) + 2(-1)(-3) = 0$$

implying that the system has nontrivial solutions, further implying that the vectors are linearly dependent and thus cannot form a basis for $\mathbb{R}^3$.

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6. [2360/030922 (24 pts)] The following parts are unrelated.

(a) (12 pts) Find the RREF of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix}$.

(b) (12 pts) We need to solve the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. After a number of elementary row operations, the augmented matrix for the system is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 5 \\ 0 & 1 & 3 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

i. (10 pts) Use this and the Nonhomogeneous Principle to find the solution to the original system.

ii. (2 pts) Find the dimension of the solution space of the original associated homogeneous system, $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Hint: You have the information you need from part (i); very little additional work is required.

SOLUTION:

(a)

$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix} \begin{array}{c} R_2^*=-1R_1+R_2 \\ R_3^*=-3R_1+R_3 \end{array} \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{bmatrix} \begin{array}{c} R_3^*=R_2+R_3 \\ R_2^*=-\frac{1}{2}R_2 \end{array} \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1^*=-3R_2+R_1 \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) i. Pivot columns correspond to x_1, x_2, x_4 so these are basic variables with x_3 and x_5, corresponding to the nonpivot columns, being free variables. Setting $x_3 = s$ and $x_5 = t$, solutions have the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5-3t \\ 4-3s+2t \\ s \\ -1+2t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{where } s,t \in \mathbb{R}$$

$$= \vec{\mathbf{x}}_p + \vec{\mathbf{x}}_h$$

ii. A basis for the solution space of the associated homogeneous system is $\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, containing two linearly independent vectors so its dimension is 2.

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7. [2360/030922 (14 pts)] Determine if the subsets, $\mathbb{W}$, are subspaces of the given vector spaces, $\mathbb{V}$.

(a) (7 pts) $\mathbb{V} = \mathbb{M}_{22}$; $\mathbb{W} = \left\{ \mathbf{A} \in \mathbb{M}_{22}, \,\middle|\, \mathbf{A}^{\mathrm{T}} = -\mathbf{A} \right\}$, the set of all matrices of the form $\begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}$ where k is a real number.

(b) (7 pts) $\mathbb{V} = \mathbb{R}^3$; $\mathbb{W} = \left\{ \vec{\mathbf{v}} \in \mathbb{R}^3 \,\middle|\, \vec{\mathbf{v}} = \begin{bmatrix} p+q \\ r \\ s \end{bmatrix} \text{ where } p,q,r,s \in \mathbb{R} \text{ and } s \geq 0 \right\}$

SOLUTION:

(a) Clearly $\vec{\mathbf{0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{W}$. Let $\vec{\mathbf{u}} = \begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix} \in \mathbb{W}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 0 & v \\ -v & 0 \end{bmatrix} \in \mathbb{W}$ and $p, q \in \mathbb{R}$. Then

$$p\vec{\mathbf{u}} + q\vec{\mathbf{v}} = p\begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix} + q\begin{bmatrix} 0 & v \\ -v & 0 \end{bmatrix} = \begin{bmatrix} 0 & pu \\ -pu & 0 \end{bmatrix} + \begin{bmatrix} 0 & qv \\ -qv & 0 \end{bmatrix} = \begin{bmatrix} 0 & pu+qv \\ -pu-qv & 0 \end{bmatrix} = \begin{bmatrix} 0 & pu+qv \\ -(pu+qv) & 0 \end{bmatrix} \in \mathbb{W}$$

since

$$\begin{bmatrix} 0 & pu+qv \\ -(pu+qv) & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & -(pu+qv) \\ pu+qv & 0 \end{bmatrix} = -\begin{bmatrix} 0 & pu+qv \\ -(pu+qv) & 0 \end{bmatrix}$$

The set is closed under linear combinations and thus is a subspace.

Alternatively, let $\mathbf{A}, \mathbf{B} \in \mathbb{W}$ and $\alpha, \beta \in \mathbb{R}$. Let $\mathbf{C} = \alpha\mathbf{A} + \beta\mathbf{B}$. Then

$$\mathbf{C}^{\mathrm{T}} = (\alpha\mathbf{A} + \beta\mathbf{B})^{\mathrm{T}} = \alpha\mathbf{A}^{\mathrm{T}} + \beta\mathbf{B}^{\mathrm{T}} = -\alpha\mathbf{A} - \beta\mathbf{B} = -(\alpha\mathbf{A} + \beta\mathbf{B}) = -\mathbf{C}.$$

Therefore $\mathbf{C} \in \mathbb{W}$, so by the Vector Subspace Theorem, $\mathbb{W}$ is a subspace of $\mathbb{V}$.

(b) Let $\vec{\mathbf{v}} = \begin{bmatrix} p+q \\ r \\ s \end{bmatrix} \in \mathbb{W}$ with $s > 0$. Then $-1\vec{\mathbf{v}} = \begin{bmatrix} -p-q \\ -r \\ -s \end{bmatrix} \notin \mathbb{W}$ since $-s < 0$. This implies that $\mathbb{W}$ is not closed under scalar multiplication and thus is not a subspace.

$\blacksquare$