- This exam is worth 100 points and has 7 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on one side.

At the top of the page containing your answer to problem 1, write the following italicized statement and sign your name to it: "*I will abide by the CU Boulder Honor Code on this exam.*" **FAILURE TO INCLUDE THIS MAY RESULT IN A PENALTY.**

1. [2360/030922 (10 pts)] Given the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ -1 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -1 & 4 \end{bmatrix}$$

write the word **TRUE** or **FALSE** as appropriate. No work need be shown, no work will be graded and no partial credit will be given.

(a) $\mathbf{CB} = \begin{bmatrix} -2 \\ 5 \\ 11 \end{bmatrix}$ (b) $\text{Tr}\left(\mathbf{B}^{\text{T}}\mathbf{A}^{\text{T}}\right) = 2$ (c) $\mathbf{A}^{\text{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\text{T}}$ (d) $|\mathbf{C}^{\text{T}}\mathbf{C} - 3\mathbf{I}| = -10$ (e) $\mathbf{AB} - \mathbf{A}^{\text{T}}\mathbf{B}^{\text{T}}$ is not defined

2. [2360/030922 (12 pts)] Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

   (a) (4 pts) Find the eigenvalues of $\mathbf{A}$ and state the multiplicity (also known as the algebraic multiplicity) of each.

   (b) (8 pts) Find the dimension of and a basis for the eigenspace associated with the eigenvalue whose (algebraic) multiplicity is greater than 1.

3. [2360/030922 (14 pts)] Let $\vec{\mathbf{p}}_1 = 1 + x^2, \vec{\mathbf{p}}_2 = x - x^2, \vec{\mathbf{p}}_3 = 2 + 2x + 4x^2$. Show that $\vec{\mathbf{p}} = 3 + 4x - 2x^2$ is in span $\{\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3\}$ by writing $\vec{\mathbf{p}}$ as a linear combination of $\vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2, \vec{\mathbf{p}}_3$. Use Cramer's Rule and cofactor expansion to solve an appropriate linear system.

4. [2360/030922 (14 pts)] Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. NO credit will be given if Gauss-Jordan elimination is used.

   (a) (5 pts) Using only matrix multiplication, verify that $\mathbf{B} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix}$ is the inverse of $\mathbf{A}$.

   (b) (9 pts) Using only matrix multiplication and properties of the matrix inverse and transpose, solve $\mathbf{A}^{\text{T}}\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

5. [2360/030922 (12 pts)] Determine if each of the following sets of vectors forms a basis for $\mathbb{R}^3$. Justify your answers.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ -2 \end{bmatrix} \right\}$

CONTINUED - MORE PROBLEMS BELOW/ON REVERSE

6. [2360/030922 (24 pts)] The following parts are unrelated.

(a) (12 pts) Find the RREF of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix}$.

(b) (12 pts) We need to solve the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. After a number of elementary row operations, the augmented matrix for the system is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 5 \\ 0 & 1 & 3 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

i. (10 pts) Use this and the Nonhomogeneous Principle to find the solution to the original system.

ii. (2 pts) Find the dimension of the solution space of the original associated homogeneous system, $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Hint: You have the information you need from part (i); very little additional work is required.

7. [2360/030922 (14 pts)] Determine if the subsets, $\mathbb{W}$, are subspaces of the given vector spaces, $\mathbb{V}$.

(a) (7 pts) $\mathbb{V} = \mathbb{M}_{22}$; $\mathbb{W} = \left\{\mathbf{A} \in \mathbb{M}_{22}, \,\middle|\, \mathbf{A}^{\mathrm{T}} = -\mathbf{A}\right\}$, the set of all matrices of the form $\begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}$ where k is a real number.

(b) (7 pts) $\mathbb{V} = \mathbb{R}^3$; $\mathbb{W} = \left\{ \vec{\mathbf{v}} \in \mathbb{R}^3 \,\middle|\, \vec{\mathbf{v}} = \begin{bmatrix} p+q \\ r \\ s \end{bmatrix} \text{ where } p, q, r, s \in \mathbb{R} \text{ and } s \geq 0 \right\}$