1. [2360/020922 (28 pts)] Consider the initial value problem $y' + y = e^t$, $y(0) = 1$.

   (a) (2 pts) Write all the numbers that apply when classifying the differential equation:
   i. separable   ii. linear   iii. autonomous   iv. homogeneous

   (b) (8 pts) Use Euler's Method to approximate $y(\ln 4) = y(2\ln 2)$ using a stepsize of $h = \ln 2$.

   (c) (6 pts) Draw the isoclines where the slope of the solution curve is $1, -1, 0$. Include line segments on each isocline indicating the slope. Label any intercepts.

   (d) (12 pts) Use the Euler-Lagrange Two Stage method (variation of parameters) to solve the initial value problem.

SOLUTION:

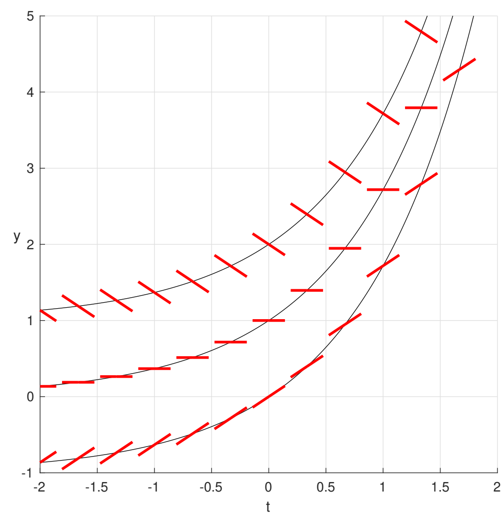
(a) ii

(b)

$$y(\ln 2) \approx y_1 = y_0 + hf(t_0, y_0) = 1 + \ln 2(e^0 - 1) = 1$$

$$y(\ln 4) = y(2\ln 2) \approx y_2 = y_1 + hf(t_1, y_1) = 1 + \ln 2(e^{\ln 2} - 1) = 1 + \ln 2$$

(c) The isoclines are $y = e^t - c$ where $c = -1, 0, 1$.



(d) Solve the associated homogeneous problem $\left(\frac{\mathrm{d}y_h}{\mathrm{d}t} + y_h = 0\right)$ using separation of variables.

$$\int \frac{\mathrm{d}y_h}{y_h} = -\int \mathrm{d}t$$

$$\ln|y_h| = -t + k$$

$$y_h(t) = Ce^{-t}$$

Set $y_p = v(t)y_h(t) = v(t)e^{-t}$ and substitute into the original nonhomogeneous equation.

$$y_p' + y_p = -v(t)e^{-t} + v'(t)e^{-t} + v(t)e^{-t} = e^t$$

$$\int v'(t)\,\mathrm{d}t = \int e^{2t}\,\mathrm{d}t$$

$$v(t) = \frac{1}{2}e^{2t}$$

$$\implies y_p(t) = \frac{1}{2}e^t$$

By the Nonhomogeneous Principle the general solution is $y(t) = y_h(t) + y_p(t) = Ce^{-t} + \frac{1}{2}e^t$ to which we apply the initial condition, giving $1 = c + \frac{1}{2} \implies c = \frac{1}{2}$ and the solution to the initial value problem as $y(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$. [or $y(t) = \cosh t$]

2. [2360/020922 (19 pts)] Consider the differential equation $y' + y^2 - 2y + 1 = 0$.

(a) (2 pts) Write all the numbers that apply when classifying the differential equation:

i. separable   ii. linear   iii. autonomous   iv. homogeneous

(b) (3 pts) Find all equilibrium solutions of the differential equation.

(c) (3 pts) Draw the phase line for the differential equation.

(d) (2 pts) Determine the stability of all equilibrium solutions.

(e) (4 pts) Show that $y = 1 + (c + t)^{-1}$ is a solution to the differential equation.

(f) (5 pts) The differential equation can be interpreted as a model of logistic growth with harvesting, where $y > 0$ is the number of fish (in thousands) and t is measured in years. If the initial $(t = 0)$ number of fish is 500, will the fish go extinct? If so, when? If not, explain why not.

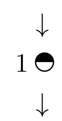
SOLUTION:

(a) i, iii

(b)

$$y' = -y^2 + 2y - 1 = -(y^2 - 2y + 1) = -(y-1)^2 \implies y = 1 \text{ is the only equilibrium solution}$$

(c)



(d) $y = 1$ is semistable.

(e)

$$\begin{aligned} y' + y^2 - 2y + 1 &= -(c+t)^{-2} + \left[1 + (c+t)^{-1}\right]^2 - 2\left[1 + (c+t)^{-1}\right] + 1 \\ &= -(c+t)^{-2} + 1 + 2(c+t)^{-1} + (c+t)^{-2} - 2 - 2(c+t)^{-1} + 1 = 0 \end{aligned}$$

(f) Apply the initial condition $\left(0, \frac{1}{2}\right)$ to give

$$\begin{aligned} \frac{1}{2} &= 1 + (c+0)^{-1} \\ -\frac{1}{2} &= \frac{1}{c} \\ c &= -2 \\ \implies y(t) &= 1 + \frac{1}{t-2} \end{aligned}$$

Setting this equal to 0 we have

$$\begin{aligned} 0 &= 1 + \frac{1}{t-2} \\ -1 &= \frac{1}{t-2} \\ t - 2 &= -1 \\ t &= 1 \end{aligned}$$

The fish will go extinct in 1 year.

3. [2360/020922 (8 pts)] Consider the initial value problem (IVP) $y' + t(y+1)^{1/3} = y^2 \cos t$, $y(-1) = -1$. Does Picard's Theorem guarantee that the IVP has a unique solution? Justify your answer.

SOLUTION:

Write the equation in the form

$$y' = f(t,y) = y^2\cos t - t(y+1)^{1/3}$$

Now y^2, $\cos t$, t and $(y+1)^{1/3}$ are continuous for all t and y. Thus $f(t,y)$ is continuous for all t and y and there is a rectangle containing $(-1,-1)$ wherein $f(t,y)$ is continuous. Picard's Theorem guarantees the existence of at least one solution to the IVP.

We also have

$$f_y(t,y) = 2y\cos t - \frac{t}{3}(y+1)^{-2/3}$$

which is not defined, and thus not continuous, at $y=-1$. Consequently, there is no rectangle containing $(-1,-1)$ wherein $f_y(t,y)$ is continuous. Picard's Theorem tells us nothing about whether or not the solution of the IVP is unique. ■

4. [2360/020922 (17 pts)] A 100 gallon tank is initially 70 percent full of vinegar in which 10 grams of the ferocious Spice 2360 is dissolved to make a tangy barbecue sauce. In both parts (a) and (b) set up, but **DO NOT SOLVE**, the initial value problem (IVP) that models the given scenario. Be sure to define your variables.

   (a) (7 pts) Vinegar containing 6 grams per gallon of the spice flows into the tank at 2 gallons per hour and the well mixed sauce leaves the tank at 2 gallons per hour.

   (b) (7 pts) Pure vinegar enters the tank at 4 gallons per hour and the well mixed sauce leaves the tank at 2 gallons per hour.

   (c) (3 pts) Over what time interval will the solution to part (b) be valid? You do not need to find the solution.

SOLUTION:

Let t be the time in hours, $s(t)$ be the amount (grams) of Spice 2360 in the tank at time t, and $V(t)$ the volume (gallons) of the sauce in the tank at time t.

(a) $V(t) = 70$ for all time.

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \text{rate in} - \text{rate out} = \left(6\,\frac{\text{gram}}{\text{gallon}}\right)\left(2\,\frac{\text{gallon}}{\text{hour}}\right) - \left(\frac{s}{70}\,\frac{\text{gram}}{\text{gallon}}\right)\left(2\,\frac{\text{gallon}}{\text{hour}}\right)$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{s}{35} = 12,\ s(0) = 10$$

(b) Since the inflow and outflow rates differ, the volume of sauce in the tank will be a function of time:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow rate in} - \text{flow rate out} = 4 - 2 = 2,\ V(0) = 70$$

$$\int \mathrm{d}V = \int 2\,\mathrm{d}t$$

$$V(t) = 2t + C$$

$$V(0) = 70 = 2(0) + C$$

$$V(t) = 2t + 70$$

Then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \text{rate in} - \text{rate out} = \left(0\,\frac{\text{gram}}{\text{gallon}}\right)\left(4\,\frac{\text{gallon}}{\text{hour}}\right) - \left(\frac{s}{2t+70}\,\frac{\text{gram}}{\text{gallon}}\right)\left(2\,\frac{\text{gallon}}{\text{hour}}\right)$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{s}{t+35} = 0,\ s(0) = 10$$

(c) Since the rate of change of the volume of sauce in the tank is positive, the tank will eventually fill up, after which the solution will no longer be valid. To see when this happens

$$V(t) = 100 = 2t + 70$$
$$2t = 30$$
$$t = 15$$

or after 15 hours. The interval over which the solution is valid is $[0,15]$ or $0 \le t \le 15$.

■

5. [2360/020922 (16 pts)] Let $t^2x' + t(t+2)x - e^{7t} = 0$.

(a) (2 pts) Write all the numbers that apply when classifying the differential equation:

i. separable   ii. linear   iii. autonomous   iv. homogeneous

(b) (12 pts) Use the integrating factor method to find the general solution of the differential equation.

(c) (2 pts) Determine the largest interval over which the general solution is defined.

SOLUTION:

(a) ii

(b)

$$x' + \frac{t+2}{t}x = \frac{e^{7t}}{t^2}$$

$$\int \frac{t+2}{t}\,\mathrm{d}t = \int \left(1 + \frac{2}{t}\right)\mathrm{d}t = t + 2\ln|t| \implies \mu(t) = e^{t+\ln t^2} = t^2e^t$$

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left(t^2e^tx\right)\mathrm{d}t = \int e^{8t}\,\mathrm{d}t$$

$$t^2e^tx(t) = \frac{1}{8}e^{8t} + C$$

$$x(t) = \frac{e^{7t}}{8t^2} + \frac{C}{t^2e^t}$$

(c) $(-\infty, 0)$ or $(0, \infty)$

6. [2360/020922 (12 pts)] Consider the system of differential equations

$$x' = 4 - x^2 - y$$

$$y' = -8 + 2x^2 - y$$

(a) On the same graph,

i. (3 pts) Plot the v nullclines, if any, as solid curve(s). Label all intercepts

ii. (3 pts) Plot the h nullclines, if any, as dashed curve(s). Label all intercepts.

iii. (2 pts) Draw an arrow (an element of the vector field) indicating the direction of the trajectory at the point $(0, 4)$.

iv. (2 pts) Draw an arrow (an element of the vector field) indicating the direction of the trajectory at the point $(0, -8)$.

(b) (2 pts) Find all equilibrium solutions, if any.

SOLUTION:

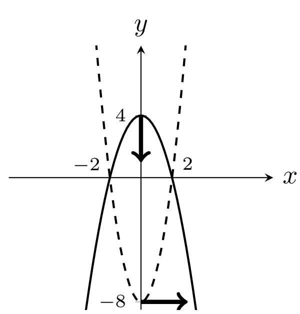
(a) i. v nullclines occur where $x' = 4 - x^2 - y = 0 \implies y = 4 - x^2$

ii. h nullclines occur where $y' = -8 + 2x^2 - y = 0 \implies y = 2x^2 - 8$.

The vector field is given by $\mathbf{V}(x, y) = \langle 4 - x^2 - y, -8 + 2x^2 - y\rangle$.

iii. $\mathbf{V}(0, 4) = \langle 0, -12\rangle$.

iv. $\mathbf{V}(0, -8) = \langle 12, 0\rangle$



(b) From the graph, h and v nullclines intersect at $(\pm 2, 0)$ so these are the equilibrium points. Or, solving the system $x' = 0, y' = 0$

$$x' = 4 - x^2 - y = 0$$
$$y' = -8 + 2x^2 - y = 0$$

$$4 - x^2 - y = -8 + 2x^2 - y$$
$$12 = 3x^2$$
$$x = \pm 2$$

showing analytically that the equilibrium points are $(\pm 2, 0)$.

■