1. [APPM 2360 Exam (25 pts)] The following problems are not related. A brief table of Laplace transforms is on page 3.

   (a) (5 pts) Use step functions to write the following function as a single expression.

$$f(t) = \begin{cases} 0, & t < 0 \\ 4, & 0 \le t < 1 \\ -2t+6, & 1 \le t < 3 \\ t^2, & 3 \le t \end{cases}$$

   (b) (5 pts) Find the inverse Laplace transform of $F(s) = \dfrac{s+6}{s^2+4s+20}$.

   (c) (3 pts) Find $\mathscr{L}\{e^t \operatorname{step}(t-2)\}$

   (d) (12 pts) Solve the initial value problem $y'' + y = \operatorname{step}(t-3) + \delta(t-4)$, $y(0) = 0$, $y'(0) = 2$.

SOLUTION:

(a)

$$\begin{aligned} f(t) &= 4\operatorname{step}(t) - 4\operatorname{step}(t-1) + (-2t+6)\operatorname{step}(t-1) - (-2t+6)\operatorname{step}(t-3) + t^2\operatorname{step}(t-3) \\ &= 4\left[\operatorname{step}(t) - \operatorname{step}(t-1)\right] + (-2t+6)\left[\operatorname{step}(t-1) - \operatorname{step}(t-3)\right] + t^2\operatorname{step}(t-3) \end{aligned}$$

(b)

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{s+6}{s^2+4s+20}\right\} &= \mathscr{L}^{-1}\left\{\frac{s+6}{s^2+4s+4-4+20}\right\} = \mathscr{L}^{-1}\left\{\frac{s+6}{(s+2)^2+16}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{s+2+4}{(s+2)^2+4^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+16}\right\} + \mathscr{L}^{-1}\left\{\frac{4}{(s+2)^2+4}\right\} \\ &= e^{-2t}\left(\cos 4t + \sin 4t\right) \end{aligned}$$

(c)

$$\mathscr{L}\left\{e^t \operatorname{step}(t-2)\right\} = e^{-2s}\mathscr{L}\left\{e^{t+2}\right\} = e^{-2s}e^2\mathscr{L}\left\{e^t\right\} = \frac{e^{2(1-s)}}{s-1}$$

(d)

$$\mathscr{L}\left\{y'' + y = \operatorname{step}(t-3) + \delta(t-4)\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-3s}}{s} + e^{-4s}$$

$$\left(s^2+1\right)Y(s) = \frac{e^{-3s}}{s} + e^{-4s} + 2$$

$$Y(s) = \frac{e^{-3s}}{s(s^2+1)} + \frac{e^{-4s}}{s^2+1} + \frac{2}{s^2+1} \quad \text{(partial fractions on the first term on the right)}$$

$$\frac{1}{s\left(s^2+1\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$$\left.\begin{array}{ll} s^2: & A+B=0 \\ s^1: & C=0 \\ s^0: & A=1 \end{array}\right\} \Longrightarrow \begin{array}{l} A=1 \\ B=-1 \\ C=0 \end{array}$$

$$Y(s) = \frac{e^{-3s}}{s} - \frac{se^{-3s}}{s^2+1} + \frac{e^{-4s}}{s^2+1} + \frac{2}{s^2+1}$$

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s} - e^{-3s}\left(\frac{s}{s^2+1}\right) + e^{-4s}\left(\frac{1}{s^2+1}\right) + 2\left(\frac{1}{s^2+1}\right)\right\} \\ &= \operatorname{step}(t-3) - \cos(t-3)\operatorname{step}(t-3) + \sin(t-4)\operatorname{step}(t-4) + 2\sin t \end{aligned}$$

2. [APPM 2360 Exam (25 pts)] The following problems are not related.

(a) (15 pts) Consider the initial value problem $2y'e^{-3t} + 3e^{2y} = 0,\ y(t_0) = y_0$.

i. (5 pts) For what initial conditions, if any, does Picard's theorem guarantee the existence of a unique solution to the initial value problem?

ii. (10 pts) Find the solution of the differential equation passing through the origin. Write your answer as an explicit function of y, that is, in the form $y = \cdots$.

(b) (10 pts) Solve the differential equation $y'' - \dfrac{t}{t-1}y' + \dfrac{y}{t-1} = 2e^t(t-1),\ t > 1$, knowing that a basis for the solution space of the associated homogeneous equation is $\{t, e^t\}$.

SOLUTION:

(a) i. The differential equation can be written as $y' = -\frac{3}{2}e^{3t+2y}$ so that $f(t,y) = -\frac{3}{2}e^{3t+2y}$ and $f_y = -3e^{3t+2y}$ which are both continuous for all values of t and y. Therefore, the initial value problem has a unique solution for all possible initial conditions.

ii. The equation is separable with initial condition $y(0) = 0$.

$$2e^{-3t}\mathrm{d}y = -3e^{2y}\mathrm{d}t$$

$$e^{-2y}\mathrm{d}y = -\frac{3}{2}e^{3t}\mathrm{d}t$$

$$\int e^{-2y}\mathrm{d}y = -\frac{3}{2}\int e^{3t}\mathrm{d}t$$

$$-\frac{1}{2}e^{-2y} = -\frac{1}{2}e^{3t} + C \quad \text{(apply initial condition)}$$

$$-\frac{1}{2} = -\frac{1}{2} + C \implies C = 0$$

$$e^{-2y} = e^{3t}$$

$$-2y = 3t$$

$$y = -\frac{3}{2}t$$

(b) Use variation of parameters with $y_p = v_1 t + v_2 e^t, \quad y_1 = t, y_2 = e^t$.

$$W[t, e^t] = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix} = te^t - e^t = e^t(t-1)$$

$$v_1 = \int \frac{-e^t\,[2e^t(t-1)]}{e^t(t-1)}\,\mathrm{d}t = -2e^t$$

$$v_2 = \int \frac{t\,[2e^t(t-1)]}{e^t(t-1)}\,\mathrm{d}t = t^2$$

so that $y_p = -2te^t + t^2e^t = te^t(t-2)$ and the general solution is thus $y = c_1 t + c_2 e^t + te^t(t-2)$.

3. [APPM 2360 Exam (28 pts)] Consider a horizontally oriented harmonic oscillator consisting of a 2 kg mass attached to a spring with restoring constant of k Nt/m. The damping is controlled with a switch which, when off makes the oscillator undamped, and when on provides a damping force equal to the instantaneous velocity.

(a) (9 pts) Assume the oscillator is unforced and that the damping switch is turned on.

i. (5 pts) If the mass is initially at its equilibrium position and given a 5 m/s push to the right, set up, but **do not solve**, the initial value problem describing this physical problem.

ii. (4 pts) For what value(s), if any, of k will the oscillator pass through the equilibrium position more than once?

(b) (8 pts) Assume that the damping switch is turned off and an external force of $f(t) = 6\cos\sqrt{5}t$ is applied to the oscillator.

i. (4 pts) Write down, but **do not solve**, the differential equation describing this physical problem.

ii. (4 pts) What value(s), if any, of k will result in unbounded solutions to the resulting differential equation?

(c) (11 pts) Now suppose that the damping switch malfunctions so that the damping constant is such that the roots of the characteristic equation are -1 and -2.

i. (3 pts) Is the oscillator underdamped, overdamped, or critically damped?

ii. (8 pts) If $x(0) = 3$ and $\dot{x}(0) = 0$, find the position of the mass after 2 seconds.

SOLUTION:

(a) i. $2\ddot{x} + \dot{x} + kx = 0,\ x(0) = 0,\ \dot{x}(0) = 5$

ii. $b^2 - 4mk < 0 \implies 1 - (4)(2)k < 0 \implies k > \frac{1}{8}$

(b) i. $2\ddot{x} + kx = 6\cos\sqrt{5}t$

ii. Need $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{5} \implies \sqrt{\frac{k}{2}} = \sqrt{5} \implies k = 10$

(c) i. overdamped since the roots of the characteristic equation are real and distinct

ii. The general solution is $x(t) = c_1 e^{-t} + c_2 e^{-2t}$. Applying the initial conditions yields

$$\left.\begin{aligned} x(0) &= c_1 + c_2 = 3 \\ \dot{x}(0) &= -c_1 - 2c_2 = 0 \end{aligned}\right\} \implies c_1 = 6, c_2 = -3$$

so $x(t) = 6e^{-t} - 3e^{-2t} \implies x(2) = 6e^{-2} - 3e^{-4} = 3e^{-4}\left(2e^2 - 1\right)$

$\blacksquare$

4. [APPM 2360 Exam (16 pts)] On the paper you will submit to Gradescope, write the letters (a) through (h). Then, next to each letter, write the word TRUE or FALSE as appropriate. No justification required and no partial credit given.

(a) It is always the case that if $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices such that $\mathbf{AB} = 0$, then either $\mathbf{A}$ or $\mathbf{B}$ must be the zero matrix.

(b) If the characteristic equation of a 4×4 matrix $\mathbf{A}$ is $4\lambda^4 + 2\lambda^2 + \lambda = 0$ then the matrix is invertible.

(c) A underdetermined (more variables that equations) system of homogeneous linear equations always has nontrivial (nonzero) solutions.

(d) Suppose $7\vec{\mathbf{u}} - 4\vec{\mathbf{v}} + 3\vec{\mathbf{w}} + \vec{\mathbf{x}} = \vec{\mathbf{0}}$ where $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}, \vec{\mathbf{x}}$ are vectors in $\mathbb{R}^4$ and $\vec{\mathbf{0}}$ is the zero vector in $\mathbb{R}^4$. Then the set $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}, \vec{\mathbf{x}}\}$ is linearly dependent.

(e) All vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $b = a + c$, is a subspace of $\mathbb{R}^3$.

(f) Any set of 6 vectors in $\mathbb{P}_5$ will span $\mathbb{P}_5$.

(g) The following system of differential equations has no equilibrium points.

$$\begin{aligned} x' &= y^4 + 1 \\ y' &= 1 + x^2 \end{aligned}$$

(h) The isoclines of the differential equation $y' + 3y^2 = 2t$ are parabolas.

SOLUTION:

(a) **FALSE**

(b) **FALSE**

(c) **TRUE**

(d) **TRUE**

(e) **TRUE**

(f) **FALSE**

(g) **TRUE**

(h) **TRUE**

$\blacksquare$

5. [APPM 2360 Exam (30 pts)] The following problems are not related.

(a) (8 pts) Use Gauss-Jordan elimination to find the matrix inverse and use the inverse to find the solution of

$$\begin{aligned} x_1 \qquad\;\; + 2x_3 &= 1 \\ x_2 - 2x_3 &= -1 \\ -2x_1 - 2x_2 + x_3 &= -3 \end{aligned}$$

Hint: your answer will involve no fractions.

(b) (8 pts) Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 4 & 2 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

i. (4 pts) Compute $\mathbf{A}^{\mathrm{T}}(\mathbf{B} - 2\mathbf{I})$.

ii. (4 pts) Compute $|\mathbf{B}|$.

(c) (14 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Write your answer as a single vector.

SOLUTION:

(a) The system can be written in the form $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ as $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3^* = 2R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -2 & 5 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3^* = 2R_2 + R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right] \xrightarrow[R_1^* = -2R_3 + R_1]{R_2^* = 2R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -4 & -2 \\ 0 & 1 & 0 & 4 & 5 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right] \implies \mathbf{A}^{-1} = \begin{bmatrix} -3 & -4 & -2 \\ 4 & 5 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 & -4 & -2 \\ 4 & 5 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ -3 \end{bmatrix}$$

(b) i.

$$\mathbf{A}^{\mathrm{T}}(\mathbf{B} - 2\mathbf{I}) = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 5 & 4 & 2 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

ii.

$$|\mathbf{B}| = 5(-1)^{1+1} \begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} + 1(-1)^{2+1} \begin{vmatrix} 4 & 2 \\ 3 & 4 \end{vmatrix} + 2(-1)^{3+1} \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 5(12-3) - 1(16-6) + 2(4-6) = 31$$

(c) Since $\mathbf{A}$ is lower triangular, the eigenvalues lie along the diagonal and are $\pm\frac{1}{2}$.

$$\lambda_1 = \frac{1}{2} : \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -\frac{1}{2} : \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The general solution is $\vec{\mathbf{x}}(t) = c_1 e^{t/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t/2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Applying the initial condition gives

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 + c_2 \end{bmatrix} \implies \begin{aligned} c_1 &= 3 \\ c_2 &= 2 \end{aligned}$$

so that the solution to the initial value problem is $\vec{\mathbf{x}}(t) = \begin{bmatrix} 3e^{t/2} \\ 3e^{t/2} + 2e^{-t/2} \end{bmatrix}$.

6. [APPM 2360 Exam (26 pts)] The following problems are not related.

(a) (8 pts) Two tanks are each partially filled with 100 gallons of brine (water in which salt is dissolved). Initially, 100 pounds of salt are dissolved in tank 1 and 50 pounds of salt are dissolved in tank 2. The system is closed in that the well-stirred liquid is pumped only between the tanks with a flow rate of 3 gallons per minute from tank 2 into tank 1 and a flow rate of 2 gallons per minute from tank 1 into tank 2. Set up, but **do not solve** the initial value problem that describes this situation. Be sure to define your variables and write your final answer in terms of matrices.

(b) (8 pts) Use the integrating factor method to find the solution of the differential equation $t^7y' + 3t^6y - 12t^6 = 0, t > 1$ satisfying $y = 0$ when $t = 1$. Identify the transient and steady state solutions.

(c) (10 pts) Consider the system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} p & 1 \\ p & p \end{bmatrix}$ and p is a parameter.

i. (2 pts) Find Tr $\mathbf{A}$.

ii. (2 pts) Find $|\mathbf{A}|$.

iii. (2 pts) Find the nonzero values, if any, of p, such that the system will have nonisolated equilibrium points.

iv. (4 pts) Classify the geometry and stability properties of the system for the following values of p.

A. (2 pts) $0 < p < 1$

B. (2 pts) $p = -1$

SOLUTION:

(a) Let $x_1(t)$ and $x_2(t)$ be the amount of salt, in pounds, of salt in tank 1 and tank 2, respectively. Likewise, let $V_1(t)$ and $V_2(t)$ represent the volumes, respectively, of brine in tank 1 and tank 2.

$$V_1' = \text{flow in} - \text{flow out} = 3 - 2 = 1, V_1(0) = 100 \implies V_1(t) = t + 100$$

$$V_2' = \text{flow in} - \text{flow out} = 2 - 3 = -1, V_2(0) = 100 \implies V_2(t) = -t + 100$$

$$x_1' = \text{rate in} - \text{rate out} = \frac{3x_2}{100 - t} - \frac{2x_1}{100 + t}$$

$$x_2' = \text{rate in} - \text{rate out} = \frac{2x_1}{100 + t} - \frac{3x_2}{100 - t}$$

Written in terms of matrices we have

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{100+t} & \dfrac{3}{100-t} \\ \dfrac{2}{100+t} & -\dfrac{3}{100-t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$$

(b) Rewrite the differential equation as $y' + \frac{3}{t}y = \frac{12}{t}$. Then the integrating factor is

$$\mu(t) = e^{\int \frac{3}{t}\,\mathrm{d}t} = t^3 \quad \text{absolute value not needed since } t > 1$$

Then

$$\int \left(t^3 y\right)' \,\mathrm{d}t = \int 12t^2 \,\mathrm{d}t$$

$$t^3 y = 4t^3 + C$$

$$y = 4 + Ct^{-3} \quad \text{apply initial condition } y(1) = 0 \implies C = -4$$

$$y = 4 - 4t^{-3}$$

The transient solution is $-4t^{-3}$ and the steady state solution is 4.

(c) i. Tr $\mathbf{A} = 2p$

ii. $|\mathbf{A}| = p^2 - p$

iii. For nonisolated equilibrium points we need $|\mathbf{A}| = p(p-1) = 0 \implies p = 0, 1$. Thus $p = 1$.

iv. A. If $0 < p < 1$ then $|\mathrm{A}| < 0$ implying that the equilibrium point $(0, 0)$ is a saddle which is unstable.

B. If $p = -1$ then Tr $\mathbf{A} = -2, |\mathrm{A}| = 2$ and $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = -4$ making the equilibrium point $(0, 0)$ an attracting spiral which is stable.