1. [APPM 2360 Exam (16 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & -10 \\ 1 & 0 & 1 \end{bmatrix}$.

   (a) (4 pts) By computing $\mathbf{A}\vec{\mathbf{v}}$, determine if $\vec{\mathbf{v}} = \begin{bmatrix} -3 \\ 12 \\ -3 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$. If it is, find the associated eigenvalue. If not, explain why not.

   (b) (4 pts) Find the characteristic equation of the matrix $\mathbf{A}$. Simplify your answer.

   (c) (4 pts) Find the eigenvalues of the matrix and state the (algebraic) multiplicity of each.

   (d) (4 pts) Find a basis for and the dimension of the eigenspace corresponding to the eigenvalue that has (algebraic) multiplicity greater than one.

SOLUTION:

(a)

$$\mathbf{A}\vec{\mathbf{v}} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & -10 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 12 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 24 \\ -6 \end{bmatrix} = 2\begin{bmatrix} -3 \\ 12 \\ -3 \end{bmatrix} = 2\vec{\mathbf{v}}$$

implying that $\vec{\mathbf{v}}$ is indeed an eigenvector with eigenvalue 2.

(b)

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -\lambda & -10 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(-\lambda) - (-\lambda) = \lambda\left[1 - \left(1 - 2\lambda + \lambda^2\right)\right] = \lambda\left(2\lambda - \lambda^2\right) = \lambda^2(2-\lambda) = 0$$

(c) $\lambda = 2$, algebraic multiplicity 1, $\lambda = 0$, algebraic multiplicity 2

(d) Need to find nontrivial solutions to $(\mathbf{A} - 0\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$. Thus

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & -10 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 = 0 \\ v_2 = t \\ v_3 = 0 \end{array}$$

and a basis for the eigenspace is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ which has dimension 1.

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2. [APPM 2360 Exam (20 pts)] Consider the differential equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \dfrac{2}{x^2}y = \dfrac{9}{x^3}$, $x > 0$.

   (a) (6 pts) Solve the associated homogeneous equation by assuming solutions of the form $y = x^r$, where r is a constant.

   (b) (10 pts) Find a particular solution of the differential equation.

   (c) (4 pts) Write the general solution of the differential equation. Simplify your answer.

SOLUTION:

(a) With $y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$, substitute into the homogeneous equation

$$y'' - \frac{2}{x^2}y = r(r-1)x^{r-2} - \frac{2}{x^2}x^r = x^{r-2}\left(r^2 - r - 2\right) = 0 \implies (r-2)(r+1) = 0 \implies r = -1, 2$$

so the solution to the homogeneous equation is $y_h(x) = c_1 x^{-1} + c_2 x^2$

(b) We must use variation of parameters so $y_p = v_1(x)y_1 + v_2(x)y_2$ with $y_1 = x^{-1}$ and $y_2 = x^2$.

$$W[y_1, y_2] = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3$$

$$v_1' = -\frac{y_2 f}{W[y_1, y_2]} = -\frac{x^2\left(\frac{9}{x^3}\right)}{3} = -\frac{3}{x} \implies v_1(x) = -\int \frac{3}{x}\mathrm{d}x = -3\ln|x| = -3\ln x \quad \text{since } x > 0$$

$$v_2' = \frac{y_1 f}{W[y_1, y_2]} = \frac{x^{-1}\left(\frac{9}{x^3}\right)}{3} = \frac{3}{x^4} \implies v_2(x) = \int 3x^{-4}\,\mathrm{d}x = -x^{-3}$$

Thus $y_p = -3\ln x\left(x^{-1}\right) + \left(-x^{-3}\right)x^2 = -3x^{-1}\ln x - x^{-1}$

(c) Using the nonhomogeneous principle,

$$y = y_h + y_p = c_1x^{-1} + c_2x^2 - 3x^{-1}\ln x - x^{-1} = c_1x^{-1} + c_2x^2 - 3x^{-1}\ln x$$

since the last term can be absorbed into the first one.

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3. [APPM 2360 Exam (24 pts)] The following problems are not related.

(a) (8 pts) Do the functions x and e^x form a basis for the solution space of the differential equation $(x-1)y'' - xy' + y = 0$ on the interval $x > 1$? Justify your answer completely.

(b) (10 pts) The characteristic equation for a certain linear, constant coefficient, homogeneous differential equation is

$$r^2(r+5)(r^2 - 8r + 25) = 0$$

Find a basis for the solution space, assuming the independent variable in the differential equation is t.

(c) (6 pts) Convert the initial value problem $3y''' - 9y'' + 18y = 0,\ y(0) = b_1, y'(0) = b_2, y''(0) = b_3$ into a system of first order differential equations, writing your answer using matrices if possible.

SOLUTION:

(a) Need to check that the functions are solutions to the differential equation.

$$(x-1)\,(x)'' - x\,(x)' + x = (x-1)(0) + x(1) + x = 0 \quad \checkmark$$
$$(x-1)\,(e^x)'' - x\,(e^x)' + x = (x-1)e^x - xe^x + e^x = xe^x - x^x - xe^x + e^x = 0 \quad \checkmark$$

Check that the functions are linearly independent.

$$W[x, e^x] = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1) \neq 0 \text{ for } x > 1 \implies \text{ functions are linearly independent}$$

The functions solve the equation and are linearly independent. Since this is a second order linear, homogeneous differential equation, the dimension of the solution space is 2 and we have two linearly independent solutions. Therefore, the functions x and e^x form a basis for the solution space.

(b) The roots of the characteristic equation are 0 with multiplicity 2, -5 with multiplicity 1, and from the quadratic formula, $4 \pm 3i$. A basis for the solution space is

$$\left\{1, t, e^{-5t}, e^{4t}\cos 3t, e^{4t}\sin 3t\right\}$$

(c) With $x_1 = y,\ x_2 = y',\ x_3 = y''$, we have $x_1(0) = y(0) = b_1, x_2(0) = y'(0) = b_2, x_3(0) = y''(0) = b_3$ and

$$x_1' = y' = x_2$$
$$x_2' = y'' = x_3$$
$$x_3' = y''' = -6y + 3y'' = -6x_1 + 3x_3$$

and written using matrices

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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4. [APPM 2360 Exam (20 pts)] Consider the differential equation $2y'' + 4y' = f(t)$.

(a) (12 pts) Use the method of undetermined coefficients to find a particular solution to the differential equation with $f(t) = 32e^{2t} - 12$ and write down the differential equation's general solution.

(b) (8 pts) For the following expressions of $f(t)$, write the FORM of the particular solution you would use to solve the differential equation using the method of undetermined coefficients. Do not find the coefficients. No partial credit will be given.

i. $f(t) = 8e^{-2t}$

ii. $f(t) = t^2e^{-t}$

iii. $f(t) = \sin 3t + \cos 2t$

iv. $f(t) = 3t^3 + t - 7$

SOLUTION:

(a) Solve the associated homogeneous equation, $2y_h'' + 4y_h' = 0$ with characteristic equation $2r^2 + 4r = 2r(r+2) = 0$. Solutions of this are $r = 0, -2$ so that $y_h = c_1 + c_2 e^{-2t}$. The particular solution has the form $y_p = Ae^{2t} + Bt$, the t being necessary since a constant is a solution of the homogeneous equation. Then $y_p' = 2Ae^{2t} + B$ and $y_p'' = 4Ae^{2t}$ and

$$2y_p'' + 4y_p' = 8Ae^{2t} + 8Ae^{2t} + 4B = 32e^{2t} - 12 \implies A = 2, B = -3 \implies y_p = 2e^{2t} - 3t$$

so that the general solution is $y = y_h + y_p = c_1 + c_2 e^{-2t} + 2e^{2t} - 3t$.

(b) i. $y_p = Ate^{-2t}$
ii. $y_p = \left(At^2 + Bt + C\right) e^{-t}$
iii. $y_p = A \sin 3t + B \cos 3t + C \sin 2t + D \cos 2t$
iv. $y_p = t\left(At^3 + Bt^2 + Ct + D\right)$

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5. [APPM 2360 Exam (20 pts)] An harmonic oscillator consists of a 0.5 kg mass attached to a spring with a restoring constant of 18 N/m. The entire apparatus is oriented horizontally and is attached to a mechanism that can turn on and off a damping force with damping constant b.

(a) (4 pts) For what value(s) of the damping constant will the oscillator be underdamped?

(b) (4 pts) Suppose the oscillator is driven by the forcing function $f(t) = \cos(2\pi t)$. Could it potentially experience resonance? Briefly explain.

(c) (12 pts) Now assume the oscillator is set up so that its governing differential equation is $\frac{1}{2}\ddot{x} + 6\dot{x} + 18x = 0$, where $x(t)$ is the displacement of the mass at time t.

i. (8 pts) Find the displacement as a function of time if, at $t = 0$ seconds, the mass is displaced 2 meters to the right of its equilibrium position and imparted a velocity of 15 meters per second towards the equilibrium position.

ii. (4 pts) Will the mass pass through its equilibrium position? If so, when? If not, explain why not.

SOLUTION:

(a) To be underdamped, we need $b^2 - 4mk < 0$ or $0 < b < \sqrt{4mk} \implies 0 < b < \sqrt{4(0.5)(18)} \implies 0 < b < 6$.

(b) No. The circular frequency of the oscillator is $\omega_0 = \sqrt{18/0.5} = 6$. Since this differs from the frequency of the forcing function, $\omega_f = 2\pi$, the oscillator will not experience resonance.

(c) We need to solve the initial value problem $\frac{1}{2}\ddot{x} + 6\dot{x} + 18x = 0$, $x(0) = 2$, $\dot{x}(0) = -15$.

i. The characteristic equation is $\frac{1}{2}r^2 + 6r + 18 = 0$ or $r^2 + 12r + 36 = (r+6)^2 = 0$ which has $r = -6$ with multiplicity 2. The general solution is thus $x(t) = c_1 e^{-6t} + c_2 t e^{-6t}$. Applying the initial conditions gives

$$x(0) = c_1 + c_2(0) = 2 \implies c_1 = 2$$
$$x(t) = e^{-6t}\left(2 + c_2 t\right)$$
$$\dot{x}(t) = c_2 e^{-6t} - 6e^{-6t}\left(2 + c_2 t\right) \implies \dot{x}(0) = c_2 - 12 = -15 \implies c_2 = -3$$

so that the displacement is $x(t) = e^{-6t}(2 - 3t)$.

ii. Passing through the equilibrium position requires that $x(t) = e^{-6t}(2 - 3t) = 0 \implies t = \frac{2}{3}$. The mass will pass through the equilibrium position at $\frac{2}{3}$ seconds.

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