1. [APPM 2360 (32 pts)] The following problems are not related.

   (a) (16 pts) For each of the following differential equations determine the order, whether or not they are linear or nonlinear, and, as necessary, homogeneous or nonhomogeneous, and constant or variable coefficient.

   i. \( y - t + 4ty' = 0 \)  
   ii. \( x'' = \sqrt{1 + (x')^2} \)  
   iii. \( y''' - 2y'' + 7y = 0 \)  
   iv. \( \frac{dy}{dx} - \frac{1}{x}y + y^2 = -\frac{4}{x^2} \)

   (b) (16 pts) Consider the initial value problem \( ty' + 2y = 4, \ y(1) = 6 \).

   i. (14 pts) Use the Euler-Lagrange Two Stage method (variation of parameters) to solve the IVP. Be sure to explicitly show all the steps in the solution process; do not simply plug into a formula.

   ii. (2 pts) Find the interval on which the solution is valid.

2. [APPM 2360 (10 pts)] The population, \( P \), of fish in a lake grows logistically and is harvested at a constant rate. The differential equation describing the population is \( \frac{dP}{dt} = P(8 - P) - 12 \) where \( t \) is in years and \( P \) is in hundreds of fish. Use the direction field below to answer the following questions.

   (a) (3 pts) For what initial populations will the fish go extinct in a finite number of years?

   (b) (7 pts) Are there any initial fish populations that will sustain themselves after a long time? If so, find them and determine the number of fish that will exist in these situations. If not, explain why not.

   ![Direction Field](image)

3. [APPM 2360 Exam (16 pts)] The following problems are not related.

   (a) (8 pts) Given the differential equation \( y' = \frac{y}{t^2 + 1} \), draw the isoclines corresponding to slopes of \(-1\) and \(1\). Be sure to include the line segments showing the slope on each isocline and label any intercepts of the isoclines and the axes.

   (b) (8 pts) On your exam paper, write (A), (B), (C) and (D). Next to each letter, write the number and the actual differential equation that corresponds to each direction field. There are more differential equations than there are direction fields so not all of the DEs will be used. You need not show any work and no partial credit will be given.

   \[
   \text{i) } y' = t^2 + y^2 \\
   \text{ii) } y' = -ty \\
   \text{iii) } y' = y^3 \cos t \\
   \text{iv) } y' = t \sin(\pi y) \\
   \text{v) } y' = t + y^4 \\
   \text{vi) } y' = -t/y
   \]
4. (APPM 2360 (22 pts)) The following problems are not related.

(a) (10 pts) For each of the following differential equations, find all equilibrium solutions. If the equation is autonomous, plot the phase line and determine the stability of any equilibrium solutions.

i. \( y' = t(y^2 + 1) \)

ii. \( y' = e^y(y - 1)^4(y^2 - 9)^3 \)

(b) (6 pts) Consider the initial value problem (IVP) \( y' = \frac{y^{1/3}}{\sqrt{t+4}}, \ y(t_0) = y_0 \). Answer the following questions, justifying your answer with Picard’s theorem.

i. Is a unique solution passing through the point \((-1, 0)\) guaranteed?

ii. For what values of \( t_0 \) and \( y_0 \) is the IVP guaranteed to have a unique solution?

(c) (6 pts) Use Euler’s method with a stepsize of \( h = \frac{1}{4} \) to estimate \( y(1.5) \) if \( y' = \frac{t^2}{y} \) and \( y(1) = \frac{1}{2} \).

5. (APPM 2360 (20 pts)) Brine (water in which salt is dissolved) containing 1 lb/gal of salt is poured at 1 gal/min into a tank that initially contained 100 gal of fresh water. The stirred mixture is drained off at 2 gal/min. Find the equation for the amount of salt in the tank at time \( t \). On what interval is this equation valid? Use the integrating factor method to solve this differential equation.