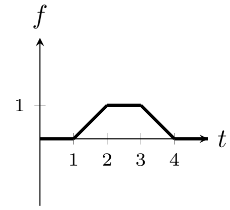
1. [25 pts] The following problems are not related.

   (a) [10 pts] Write the following function using the unit step function, that is, not as a piecewise function.



   (b) [15 pts] Solve $y'' + y = 4\delta(t - 2\pi) + \text{step}(t - e),\ y(0) = 1, y'(0) = 0.$

SOLUTION:

(a) Just for reference, here is the function defined piecewise. $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ 4-t & 3 \le t < 4 \\ 0 & 4 \le t \end{cases}$ Using the unit step function,

$$\begin{aligned} f(t) &= (t-1)\text{step}(t-1) - (t-1)\text{step}(t-2) + \text{step}(t-2) - \text{step}(t-3) + (4-t)\text{step}(t-3) - (4-t)\text{step}(t-4) \\ &= (t-1)\left[\text{step}(t-1) - \text{step}(t-2)\right] + \left[\text{step}(t-2) - \text{step}(t-3)\right] + (4-t)\left[\text{step}(t-3) - \text{step}(t-4)\right] \end{aligned}$$

(b) Taking the Laplace transform of both sides of the differential equation yields

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s} + \frac{e^{-es}}{s}$$

$$Y(s) = \frac{4e^{-2\pi s}}{s^2+1} + \frac{e^{-es}}{s\,(s^2+1)} + \frac{s}{s^2+1} \qquad \text{partial fractions}: \quad \frac{1}{s\,(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{4e^{-2\pi s}}{s^2+1}\right\} + \mathscr{L}^{-1}\left\{\frac{e^{-es}}{s} - \frac{e^{-es}s}{s^2+1}\right\} + \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\ &= 4\sin(t-2\pi)\text{step}(t-2\pi) + \text{step}(t-e) - \cos(t-e)\text{step}(t-e) + \cos t \\ &= \cos t + 4\sin t\,\text{step}(t-2\pi) + \left[1 - \cos(t-e)\right]\text{step}(t-e) \end{aligned}$$

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2. [35 pts] The following problems are not related.

   (a) [10 pts] The motion of an harmonic oscillator is governed by the differential equation $2\ddot{x} + 3\dot{x} + 4x = g(t)$.

      i. Suppose the oscillator is unforced and the motion is started from rest with an initial displacement of 5 positive units from the equilibrium position. Will the oscillator pass through the equilibrium position multiple times? Justify your answer.

      ii. Now suppose the oscillator experiences a forcing function $2e^t$ for the first two seconds, after which it is removed. Later, the oscillator is given a blow with a hammer that instantaneously imparts 3 units of force at precisely 5 seconds. Find $g(t)$ in this case. No points awarded for answers written piecewise.

   (b) [10 pts] Consider the differential equation $y''' + ay'' + by' + cy = f(t)$ where a, b and c are real constants. The solution to the associated homogeneous equation is $y_h(t) = c_1e^{-t} + e^{2t}\left(c_2\cos t + c_3\sin t\right)$. For each of the five parts below, find the form of the particular solution for the given $f(t)$ to be used in the method of undetermined coefficients. Do not solve for the coefficients and write "N/A" if the method is not applicable.

      i. $f(t) = e^t + e^{-t}$ ii. $f(t) = t^{-2}e^{2t}$ iii. $f(t) = e^{2t}\sin t + \cos 2t$ iv. $f(t) = 3t^3 - 9t$ v. $f(t) = t\ln t$

   (c) [15 pts] Consider the differential equation $ty' + 4y + 2t^{-3}\sec^2\left(\frac{\pi}{4}t\right) = 0$.

      i. [5 pts] Does Picard's Theorem guarantee the existence of a unique solution to the initial value problem consisting of the differential equation and the initial condition $y(2) = 0$? Justify your answer.

ii. [10 pts] Assuming that $t > 0$, use the integrating factor method to find the solution to the differential equation that passes through the point $\left(1, \frac{8}{\pi}\right)$. Simplify your answer. No points awarded for using any other method. [Recall $(\tan x)' = \sec^2 x$]

SOLUTION:

(a) i. We have $b^2 - 4mk = 3^2 - 4(2)(4) = -23 < 0$ implying that the oscillator is underdamped. Furthermore the roots of the characteristic equation are $\dfrac{-3 \pm \sqrt{23}i}{4}$ which are nonreal. The solution will have the form $e^{-bt/2m}(c_1 \cos \omega_d t + c_2 \sin \omega_d t)$ or $A(t)\cos(\omega_d t - \delta)$ which has an infinite number of real zeros. This implies that the oscillator will pass through the equilibrium position multiple times.

ii. The forcing function can be written several ways:

$$\begin{aligned} g(t) &= 2e^t - 2e^t\text{step}(t-2) + 3\delta(t-5) = 2e^t\left[1 - \text{step}(t-2)\right] + 3\delta(t-5) \\ &= 2e^t\text{step}(t) - 2e^t\text{step}(t-2) + 3\delta(t-5) = 2e^t\left[\text{step}(t) - \text{step}(t-2)\right] + 3\delta(t-5) \end{aligned}$$

(b) In each case we'll choose an initial form for y_p and then decide if it needs modification based on the solutions to the homogeneous problem in part (a).

i. Initial $y_p = Ae^t + Be^{-t}$ but e^{-t} is a solution to the homogeneous problem. Thus, final $y_p = Ae^t + Bte^{-t}$

ii. N/A

iii. Initial $y_p = e^{2t}(A\sin t + B\cos t) + C\cos 2t + D\sin 2t$ but $e^{2t}\sin t$ is a solution to the homogeneous equation. Thus, final $y_p = e^{2t}t(A\sin t + B\cos t) + C\cos 2t + D\sin 2t$

iv. Initial $y_p = At^3 + Bt^2 + Ct + D$. None of these are solutions to the homogeneous problem so this is our final y_p as well.

v. N/A

(c) i. To apply Picard's theorem, write the equation as $y' = -\frac{4}{t}y - 2t^{-4}\sec^2\left(\frac{\pi}{4}t\right) = f(t,y)$. Since $f(2,0)$ is not defined, $f(t,y)$ is not continuous at $(2,0)$ and Picard's theorem cannot be used to guarantee even the existence of a solution, much less a unique one.

ii. To use the integrating factor method, write the differential equation as $y' + \frac{4}{t}y = -2t^{-4}\sec^2\left(\frac{\pi}{4}t\right)$. Then

$$\mu(t) = e^{\int \frac{4}{t}\,dt} = e^{4\ln t} = t^4$$

$$\left(t^4 y\right)' = -2\sec^2\left(\frac{\pi}{4}t\right)$$

$$t^4 y = -2\int \sec^2\left(\frac{\pi}{4}t\right) = -\frac{8}{\pi}\tan\left(\frac{\pi}{4}t\right) + C \qquad \text{apply initial condition}$$

$$(1^4)\left(\frac{8}{\pi}\right) = -\frac{8}{\pi}\tan\left(\frac{\pi}{4}\right) + C \implies C = \frac{16}{\pi}$$

$$y(t) = \frac{8}{\pi}\left[\frac{2 - \tan\left(\frac{\pi}{4}t\right)}{t^4}\right]$$

3. [35 pts] The following problems are not related.

(a) [10 pts] Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ with $\mathbf{A} = \begin{bmatrix} k & 4 \\ -1 & 1 \end{bmatrix}$.

i. Find all values of k such that $\mathbf{A}$ has a repeated eigenvalue.

ii. Find all values of k such that the equilibrium solution $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a center.

iii. Find all values of k such that there is a zero eigenvalue.

iv. Describe the stability and geometry of the equilibrium solution $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if $k = 6$.

v. For what value(s) of k will the equilibrium solution(s) be stable?

(b) [15 pts] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix}\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, writing your answer as a single vector.

(c) [10 pts] Consider the matrices $\mathbf{C} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -1 & -2 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$.

i. Compute $\mathbf{CD}$.

ii. Find the solution of $\mathbf{C}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ in terms of $\mathbf{D}$.

SOLUTION:

(a) We have $\text{Tr}\,\mathbf{A} = k+1$ and $|\mathbf{A}| = k+4$.

i. To have repeated eigenvalues we need $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = (k+1)^2 - 4(k+4) = k^2 - 2k - 15 = (k+3)(k-5) = 0 \implies k = -3, 5$.

ii. For a center, we need $\text{Tr}\,\mathbf{A} = 0$ and $|\mathbf{A}| > 0$. These two conditions require $k = -1$ and $k > -4$ implying that $k = -1$.

iii. For $\mathbf{A}$ to have a zero eigenvalue requires that $|\mathbf{A}| = 0$. This will occur if $k = -4$.

iv. If $k = 6$, then $\text{Tr}\,\mathbf{A} = 7$ and $|\mathbf{A}| = 10$ so that $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 9 > 0$. This results in the equilibrium point being a repelling node which is unstable.

v. For this case we need $\text{Tr}\,\mathbf{A} \le 0 \implies k+1 \le 0$ and $|\mathbf{A}| \ge 0 \implies k+4 \ge 0$. Together these imply $k \le -1$ and $k \ge -4$ so $-4 \le k \le -1$ or $k \in [-4, -1]$.

(b)

$$\begin{vmatrix} -4-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} = (-4-\lambda)(2-\lambda) + 9 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \implies \lambda = -1 \text{ with algebraic multiplicity of 2}$$

$$(\mathbf{A}+\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}} \implies \left[\begin{array}{cc|c} -3 & 3 & 0 \\ -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Because $\lambda = -1$ has only one eigenvector, we need to find the generalized eigenvector.

$$(\mathbf{A}+\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{v}} \implies \left[\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{u}} = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}$$

This gives the general solution as $\vec{\mathbf{x}}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} \right)$. Applying the initial condition gives

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \implies c_1 = 1, c_2 = 9$$

so that the solution to the initial value problem is

$$\vec{\mathbf{x}}(t) = e^{-t} \begin{bmatrix} 9t+1 \\ 9t+4 \end{bmatrix}$$

(c) i.

$$\mathbf{CD} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -1 & -2 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

ii. Since $\mathbf{C}$ is invertible with $\mathbf{C}^{-1} = \mathbf{D}$, we have

$$\mathbf{C}^{-1}\mathbf{C}\vec{\mathbf{x}} = \mathbf{I}\vec{\mathbf{x}} = \vec{\mathbf{x}} = \mathbf{C}^{-1}\vec{\mathbf{b}} = \mathbf{D}\vec{\mathbf{b}}$$

4. [35 pts] The following problems are not related.

(a) [20 pts] Consider the system of equations

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 2 \\ 5x_1 - x_2 + x_3 - x_4 &= 6 \end{aligned}$$

i. Form the augmented matrix associated with the system and perform Gauss-Jordan elimination to find the reduced row-echelon form.

ii. Find a particular solution to the given system.

iii. Find the dimension of and a basis for the solution space of the associated homogeneous system.

iv. Write the general solution to the system.

(b) [10 pts] Consider the vectors $\vec{\mathbf{v}}_1 = [1, 1, 2]^{\mathrm{T}}$, $\vec{\mathbf{v}}_2 = [1, 0, 1]^{\mathrm{T}}$, $\vec{\mathbf{v}}_3 = [2, 1, 3]^{\mathrm{T}}$.

i. [6 pts] By evaluating an appropriate determinant, decide whether or not the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is linearly independent.

ii. [4 pts] Does span $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} = \mathbb{R}^3$? Justify your answer.

(c) [5 pts] Consider the set $\mathbb{W}$ of all real 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $b + c = 0$. Is this set a subspace of $\mathbb{M}_{22}$? Justify your answer.

SOLUTION:

(a) i.

$$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 2 \\ 5 & -1 & 1 & -1 & 6 \end{array}\right] \xrightarrow{R_2^* = R_1 + R_2} \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 2 \\ 8 & 0 & 2 & 0 & 8 \end{array}\right] \xrightarrow{R_2^* = \frac{1}{8}R_2} \left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 2 \\ 1 & 0 & \frac{1}{4} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 1 \\ 3 & 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2^* = -3R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 1 \\ 0 & 1 & \frac{1}{4} & 1 & -1 \end{array}\right]$$

ii. From the RREF matrix above, $x_3 = s$ and $x_4 = t$ are free variables. Thus

$$x_1 = 1 - \frac{1}{4}s$$
$$x_2 = -1 - \frac{1}{4}s - t$$

so choosing $s = t = 0$, a particular solution is $\vec{\mathbf{x}}_p = \begin{bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \\ x_{p_4} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

iii. Using the same RREF for the homogeneous system gives

$$\vec{\mathbf{x}}_h = \begin{bmatrix} x_{h_1} \\ x_{h_2} \\ x_{h_3} \\ x_{h_4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}s \\ -\frac{1}{4}s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the solution space of the homogeneous systems is then

$$\left\{ \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of the solution space is 2.

iv. The general solution of the system is

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = s \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

(b) i. To determine if the set of vectors is linearly independent, we need to decide if the only solution to $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ is the trivial solution. This is equivalent to

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Computing the determinant of the coefficient matrix by expanding along the second row yields

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1(-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

This implies that the homogeneous system has nontrivial solutions, further implying that the vectors are linearly dependent.

ii. The dimension of $\mathbb{R}^3$ is 3. Since this set of vectors is not linearly independent, despite the fact that there are three vectors, they cannot span $\mathbb{R}^3$.

(c) Let $\vec{\mathbf{p}} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\vec{\mathbf{q}} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be two vectors in the set $\mathbb{W}$. Then $b_1 + c_1 = 0$ and $b_2 + c_2 = 0$. With r and s real numbers, consider

$$r\vec{\mathbf{p}} + s\vec{\mathbf{q}} = \begin{bmatrix} ra_1 & rb_1 \\ rc_1 & rd_1 \end{bmatrix} + \begin{bmatrix} sa_2 & sb_2 \\ sc_2 & sd_2 \end{bmatrix} = \begin{bmatrix} ra_1 + sa_2 & rb_1 + sb_2 \\ rc_1 + sc_2 & rd_1 + sd_2 \end{bmatrix}$$

The sum of the off diagonal elements is

$$(rb_1 + sb_2) + (rc_1 + sc_2) = r(b_1 + c_1) + s(b_2 + c_2) = 0$$

showing that $r\vec{\mathbf{p}} + s\vec{\mathbf{q}}$ is also in the set $\mathbb{W}$. This implies that the set $\mathbb{W}$ is closed under addition and scalar multiplication. Therefore the set $\mathbb{W}$ is a subspace of $\mathbb{M}_{22}$.

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5. [20 pts] The following problems are not related.

(a) [10 pts] Fresh water flows into tank A at a rate of 3 gallons per minute. The well-stirred solution flows into tank B at a rate of 5 gallons per minute, but 2 gallons per minute are fed back from tank B to tank A, and an additional 3 gallons per minute drain from tank B. If tank A contains 50 gallons of solution containing 20 pounds of salt, and tank B contains 60 gallons of solution with 10 pounds of dissolved salt at the start, set up, but **DO NOT EVALUATE** an initial value problem describing this scenario. Be sure to identify your variables and write your final answer in terms of matrices and vectors.

(b) [10 pts] Find all the v and h nullclines and equilibrium solutions of the nonlinear system of differential equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y^2 - xy - 3y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x^2 + xy - x \end{aligned}$$

SOLUTION:

(a) Let $x_A(t)$ be the amount of salt in tank A at time t and $x_B(t)$ be the amount of salt in tank B at time t. Using the facts that the volume of solution in both tanks remains constant and that the rate of change of salt in the tank equals the difference between the rate of salt coming in and the rate of salt going out gives the following:

$$\begin{aligned} \frac{\mathrm{d}x_A}{\mathrm{d}t} &= \left(0\frac{\text{lb}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right) + \left(\frac{x_B}{60}\frac{\text{lb}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x_A}{50}\frac{\text{lb}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) = -\frac{1}{10}x_A + \frac{1}{30}x_B \\ \frac{\mathrm{d}x_B}{\mathrm{d}t} &= \left(\frac{x_A}{50}\frac{\text{lb}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x_B}{60}\frac{\text{lb}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x_B}{60}\frac{\text{lb}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right) = \frac{1}{10}x_A - \frac{1}{12}x_B \end{aligned}$$

In terms of matrices and vectors we have

$$\begin{bmatrix} x_A \\ x_B \end{bmatrix}' = \begin{bmatrix} -\frac{1}{10} & \frac{1}{30} \\ \frac{1}{10} & -\frac{1}{12} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix}, \quad \begin{bmatrix} x_A(0) \\ x_B(0) \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

(b)

$$\begin{aligned} v \text{ nullclines} : \frac{\mathrm{d}x}{\mathrm{d}t} = 0 &\implies y^2 - xy - 3y = y(y - x - 3) = 0 \implies y = 0,\ y = x + 3 \\ h \text{ nullclines} : \frac{\mathrm{d}y}{\mathrm{d}t} = 0 &\implies x^2 + xy - x = x(x + y - 1) = 0 \implies x = 0,\ y = 1 - x \end{aligned}$$

Equilibrium points occur where the v and h nullclines intersect, that is, where both $\mathrm{d}x/\mathrm{d}t = 0$ and $\mathrm{d}y/\mathrm{d}t = 0$. Consider the system of equations

$$y(y - x - 3) = 0 \tag{1}$$
$$x(x + y - 1) = 0 \tag{2}$$

If $y = 0$ in (1), then either $x = 0$ or $x = 1$ satifies (2). This gives the equilibrium points $(0, 0), (1, 0)$. If $y = x + 3$ in (1), then either $x = 0$ or $x + y - 1 = x + (x + 3) - 1 = 2x + 2 = 0$ satisfies (2). Then, respectively, $y = 3$ or $x = -1$, giving $(0, 3), (-1, 2)$ as two other equilibrium points.

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