1. [25 pts] The following problems are not related.
   (a) [10 pts] Write the following function using the unit step function, that is, not as a piecewise function.
   \[
   f(t) = \begin{cases} 
   1 & \text{if } 2 \leq t < 4 \\
   0 & \text{otherwise}
   \end{cases}
   \]

   (b) [15 pts] Solve \( y'' + y = 4\delta(t - 2\pi) + \text{step}(t - e) \), \( y(0) = 1, \ y'(0) = 0 \).

2. [35 pts] The following problems are not related.
   (a) [10 pts] The motion of an harmonic oscillator is governed by the differential equation \( 2\ddot{x} + 3\dot{x} + 4x = g(t) \).
      i. Suppose the oscillator is unforced and the motion is started from rest with an initial displacement of 5 positive units from the equilibrium position. Will the oscillator pass through the equilibrium position multiple times? Justify your answer.
      ii. Now suppose the oscillator experiences a forcing function \( 2e^t \) for the first two seconds, after which it is removed. Later, the oscillator is given a blow with a hammer that instantaneously imparts 3 units of force at precisely 5 seconds. Find \( g(t) \) in this case. No points awarded for answers written piecewise.

   (b) [10 pts] Consider the differential equation \( y'' + ay' + by + cy = f(t) \) where \( a, b \) and \( c \) are real constants. The solution to the associated homogeneous equation is \( y_h(t) = c_1e^{-t} + e^{2t}(c_2\cos t + c_3\sin t) \). For each of the five parts below, find the form of the particular solution for the given \( f(t) \) to be used in the method of undetermined coefficients. Do not solve for the coefficients and write \( \text{"N/A"} \) if the method is not applicable.
      i. \( f(t) = e^t + e^{-t} \)
      ii. \( f(t) = t^{-2}e^{2t} \)
      iii. \( f(t) = e^{2t}\sin t + \cos 2t \)
      iv. \( f(t) = 3t^3 - 9t \)
      v. \( f(t) = t\ln t \)

   (c) [15 pts] Consider the differential equation \( ty' + 4y + 2t^{-3}\sec^2\left(\frac{\pi}{t}\right) = 0 \).
      i. [5 pts] Does Picard’s Theorem guarantee the existence of a unique solution to the initial value problem consisting of the differential equation and the initial condition \( y(2) = 0 \)? Justify your answer.
      ii. [10 pts] Assuming that \( t > 0 \), use the integrating factor method to find the solution to the differential equation that passes through the point \((1, \frac{\pi}{\pi})\). Simplify your answer. No points awarded for using any other method. [Recall \((\tan x)' = \sec^2 x\)]

3. [35 pts] The following problems are not related.
   (a) [10 pts] Consider the system of differential equations \( \ddot{\vec{x}} = \mathbf{A}\vec{x} \) with \( \mathbf{A} = \begin{bmatrix} k & 4 \\ -1 & 1 \end{bmatrix} \).
      i. Find all values of \( k \) such that \( \mathbf{A} \) has a repeated eigenvalue.
      ii. Find all values of \( k \) such that the equilibrium solution \( \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) is a center.
      iii. Find all values of \( k \) such that there is a zero eigenvalue.
      iv. Describe the stability and geometry of the equilibrium solution \( \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) if \( k = 6 \).
      v. For what value(s) of \( k \) will the equilibrium solution(s) be stable?

   (b) [15 pts] Solve the initial value problem \( \ddot{\vec{x}} = \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix}\vec{x}, \ \vec{x}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \), writing your answer as a single vector.

   (c) [10 pts] Consider the matrices \( \mathbf{C} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \) and \( \mathbf{D} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -1 & -2 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \).
      i. Compute \( \mathbf{C}\mathbf{D} \).
      ii. Find the solution of \( \mathbf{C}\vec{x} = \mathbf{D} \) in terms of \( \mathbf{D} \).

CONTINUED
4. (35 pts) The following problems are not related.

(a) (20 pts) Consider the system of equations

\[\begin{align*}
3x_1 + x_2 + x_3 + x_4 &= 2 \\
5x_1 - x_2 + x_3 - x_4 &= 6
\end{align*}\]

i. Form the augmented matrix associated with the system and perform Gauss-Jordan elimination to find the reduced row-echelon form.

ii. Find a particular solution to the given system.

iii. Find the dimension of and a basis for the solution space of the associated homogeneous system.

iv. Write the general solution to the system.

(b) (10 pts) Consider the vectors \( \mathbf{v}_1 = [1, 1, 2]^T, \mathbf{v}_2 = [1, 0, 1]^T, \mathbf{v}_3 = [2, 1, 3]^T. \)

i. By evaluating an appropriate determinant, decide whether or not the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is linearly independent.

ii. Does \( \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} = \mathbb{R}^3 \)? Justify your answer.

(c) (5 pts) Consider the set \( \mathbb{W} \) of all real \( 2 \times 2 \) matrices \[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\] with \( b + c = 0 \). Is this set a subspace of \( \mathbb{M}_{22} \)? Justify your answer.

5. (20 pts) The following problems are not related.

(a) (10 pts) Fresh water flows into tank A at a rate of 3 gallons per minute. The well-stirred solution flows into tank B at a rate of 5 gallons per minute, but 2 gallons per minute are fed back from tank B to tank A, and an additional 3 gallons per minute drain from tank B. If tank A contains 50 gallons of solution containing 20 pounds of salt, and tank B contains 60 gallons of solution with 10 pounds of dissolved salt at the start, set up, but do not solve, an initial value problem describing this scenario.

(b) (10 pts) Find all the \( v \) and \( h \) nullclines and equilibrium solutions of the nonlinear system of differential equations

\[\begin{align*}
\frac{dx}{dt} &= y^2 - xy - 3y \\
\frac{dy}{dt} &= x^2 + xy - x
\end{align*}\]

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**Short table of Laplace Transforms:**

\[\mathcal{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t) \, dt\]

In this table, \( a, b, c \) are real numbers with \( c \geq 0 \), and \( n = 0, 1, 2, 3, \ldots \)

\[\begin{align*}
\mathcal{L}\{t^n e^{at}\} &= \frac{n!}{(s-a)^{n+1}} \\
\mathcal{L}\{e^{at}\cos bt\} &= \frac{s-a}{(s-a)^2 + b^2} \\
\mathcal{L}\{e^{at}\sin bt\} &= \frac{b}{(s-a)^2 + b^2} \\
\mathcal{L}\{t^n f(t)\} &= (-1)^n \frac{d^n F(s)}{ds^n} \\
\mathcal{L}\{e^{at} f(t)\} &= F(s-a) \\
\mathcal{L}\{\delta(t-c)\} &= e^{-cs} \\
\mathcal{L}\{tf'(t)\} &= -F(s) - s \frac{dF(s)}{ds} \\
\mathcal{L}\{f(t-c) \text{step}(t-c)\} &= e^{-cs} F(s) \\
\mathcal{L}\{f(t) \text{step}(t-c)\} &= e^{-cs} \mathcal{L}\{f(t+c)\} \\
\mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0)
\]