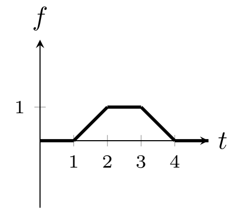
1. [25 pts] The following problems are not related.

   (a) [10 pts] Write the following function using the unit step function, that is, not as a piecewise function.



   (b) [15 pts] Solve $y'' + y = 4\delta(t - 2\pi) + \text{step}(t - e),\ y(0) = 1, y'(0) = 0.$

2. [35 pts] The following problems are not related.

   (a) [10 pts] The motion of an harmonic oscillator is governed by the differential equation $2\ddot{x} + 3\dot{x} + 4x = g(t)$.

      i. Suppose the oscillator is unforced and the motion is started from rest with an initial displacement of 5 positive units from the equilibrium position. Will the oscillator pass through the equilibrium position multiple times? Justify your answer.

      ii. Now suppose the oscillator experiences a forcing function $2e^t$ for the first two seconds, after which it is removed. Later, the oscillator is given a blow with a hammer that instantaneously imparts 3 units of force at precisely 5 seconds. Find $g(t)$ in this case. No points awarded for answers written piecewise.

   (b) [10 pts] Consider the differential equation $y''' + ay'' + by' + cy = f(t)$ where a, b and c are real constants. The solution to the associated homogeneous equation is $y_h(t) = c_1e^{-t} + e^{2t}(c_2 \cos t + c_3 \sin t)$. For each of the five parts below, find the form of the particular solution for the given $f(t)$ to be used in the method of undetermined coefficients. Do not solve for the coefficients and write "N/A" if the method is not applicable.

      i. $f(t) = e^t + e^{-t}$ ii. $f(t) = t^{-2}e^{2t}$ iii. $f(t) = e^{2t}\sin t + \cos 2t$ iv. $f(t) = 3t^3 - 9t$ v. $f(t) = t \ln t$

   (c) [15 pts] Consider the differential equation $ty' + 4y + 2t^{-3}\sec^2\left(\frac{\pi}{4}t\right) = 0$.

      i. [5 pts] Does Picard's Theorem guarantee the existence of a unique solution to the initial value problem consisting of the differential equation and the initial condition $y(2) = 0$? Justify your answer.

      ii. [10 pts] Assuming that $t > 0$, use the integrating factor method to find the solution to the differential equation that passes through the point $\left(1, \frac{8}{\pi}\right)$. Simplify your answer. No points awarded for using any other method. [Recall $(\tan x)' = \sec^2 x$]

3. [35 pts] The following problems are not related.

   (a) [10 pts] Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ with $\mathbf{A} = \begin{bmatrix} k & 4 \\ -1 & 1 \end{bmatrix}$.

      i. Find all values of k such that $\mathbf{A}$ has a repeated eigenvalue.

      ii. Find all values of k such that the equilibrium solution $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a center.

      iii. Find all values of k such that there is a zero eigenvalue.

      iv. Describe the stability and geometry of the equilibrium solution $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if $k = 6$.

      v. For what value(s) of k will the equilibrium solution(s) be stable?

   (b) [15 pts] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, writing your answer as a single vector.

   (c) [10 pts] Consider the matrices $\mathbf{C} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -1 & -2 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$.

      i. Compute $\mathbf{CD}$.

      ii. Find the solution of $\mathbf{C}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ in terms of $\mathbf{D}$.

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4. [35 pts] The following problems are not related.

(a) [20 pts] Consider the system of equations

$$3x_1 + x_2 + x_3 + x_4 = 2$$
$$5x_1 - x_2 + x_3 - x_4 = 6$$

i. Form the augmented matrix associated with the system and perform Gauss-Jordan elimination to find the reduced row-echelon form.

ii. Find a particular solution to the given system.

iii. Find the dimension of and a basis for the solution space of the associated homogeneous system.

iv. Write the general solution to the system.

(b) [10 pts] Consider the vectors $\vec{\mathbf{v}}_1 = [1, 1, 2]^{\mathrm{T}}$, $\vec{\mathbf{v}}_2 = [1, 0, 1]^{\mathrm{T}}$, $\vec{\mathbf{v}}_3 = [2, 1, 3]^{\mathrm{T}}$.

i. [6 pts] By evaluating an appropriate determinant, decide whether or not the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is linearly independent.

ii. [4 pts] Does span $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} = \mathbb{R}^3$? Justify your answer.

(c) [5 pts] Consider the set $\mathbb{W}$ of all real 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $b + c = 0$. Is this set a subspace of $\mathbb{M}_{22}$? Justify your answer.

5. [20 pts] The following problems are not related.

(a) [10 pts] Fresh water flows into tank A at a rate of 3 gallons per minute. The well-stirred solution flows into tank B at a rate of 5 gallons per minute, but 2 gallons per minute are fed back from tank B to tank A, and an additional 3 gallons per minute drain from tank B. If tank A contains 50 gallons of solution containing 20 pounds of salt, and tank B contains 60 gallons of solution with 10 pounds of dissolved salt at the start, set up, but **DO NOT EVALUATE** an initial value problem describing this scenario. Be sure to identify your variables and write your final answer in terms of matrices and vectors.

(b) [10 pts] Find all the v and h nullclines and equilibrium solutions of the nonlinear system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y^2 - xy - 3y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x^2 + xy - x$$

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\{e^{at} f(t)\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$