1. [20 pts] Let $y_1 = t^{1/2}$ and $y_2 = t^{-3/2}$ be solutions to the ODE $L(y) = y'' + a_1(t)y' + a_0(t)y = 0, \;\; t > 0$, where $a_1(t)$ and $a_0(t)$ are continuous on $[0, \infty)$.

   (a) [8 pts] Is the set $\{y_1, y_2\}$ a basis for the solution space of the ODE? Justify your answer.

   (b) [12 pts] Find the general solution of $L(y) = 4t^{5/2}$.

SOLUTION:

(a) Since we are told that the functions solve the second order ODE, and there are two of them, we only need to check that they are linearly independent, which we do using the Wronskian.

$$W[y_1, y_2](t) = \begin{vmatrix} t^{1/2} & t^{-3/2} \\ \frac{1}{2}t^{-1/2} & -\frac{3}{2}t^{-5/2} \end{vmatrix} = -2t^{-2} \neq 0$$

The nonvanishing of the Wronskian shows that the functions are linearly independent and thus they form a basis for the solution space.

(b) We use variation of parameters to find the particular solution with $y_1 = t^{1/2}$, $y_2 = t^{-3/2}$, $f(t) = 4t^{5/2}$ and $y_p = v_1y_1 + v_2y_2$.

$$v_1' = -\frac{y_2 f}{W} = -\frac{t^{-3/2}(4t^{5/2})}{-2t^{-2}} = 2t^3 \implies v_1 = \int 2t^3 \, \mathrm{d}t = \frac{1}{2}t^4$$

$$v_2' = \frac{y_1 f}{W} = \frac{t^{1/2}(4t^{5/2})}{-2t^{-2}} = -2t^5 \implies v_2 = \int -2t^5 \, \mathrm{d}t = -\frac{1}{3}t^6$$

$$y_p = \frac{1}{2}t^4t^{1/2} - \frac{1}{3}t^6t^{-3/2} = \frac{1}{6}t^{9/2}$$

$$y(t) = y_h + y_p = c_1t^{1/2} + c_2t^{-3/2} + \frac{1}{6}t^{9/2}$$

■

2. [20 pts] The following problems are not related.

   (a) [6 pts] Convert the initial value problem $y''' - ty'' + y^2y' - \sqrt{3y} = t^2 + t + 1, \;\; y(0) = -2, \; y'(0) = 0, \; y''(0) = 4$ into an initial value problem for a first order system.

   (b) [14 pts] Consider an harmonic oscillator consisting of a spring attached to a 2 kg object.

      i. [3 pts] A force equal to 10 newtons is required to stretch the spring 2 meters. Find the spring constant, assuming Hooke's Law applies.

      ii. [8 pts] The oscillator is immersed in a medium that offers a damping force equal to $\sqrt{p}$ times the instantaneous velocity. Determine the value(s) of p so that the subsequent motion is (i) underdamped, (ii) critically damped, (iii) overdamped.

      iii. [3 pts] Now suppose that the system is undamped but the motion is forced by $f(t) = \cos(\omega_f t)$. Find the value of ω_f that will put the system into resonance.

SOLUTION:

(a) Let $u_1 = y, u_2 = y', u_3 = y''$. Then $u_1' = y', u_2' = y'', u_3' = y''' = t^2 + t + 1 + \sqrt{3y} - y^2y' + ty''$. Thus

$$\begin{aligned} u_1' &= u_2 & u_1(0) &= -2 \\ u_2' &= u_3 & u_2(0) &= 0 \\ u_3' &= \sqrt{3u_1} - u_1^2u_2 + tu_3 + t^2 + t + 1 & u_3(0) &= 4 \end{aligned}$$

(b) i. Using $F = kx$ we have $k = F/x = 10/2 = 5$ newtons per meter.

ii. The differential equation describing this situation is $2\ddot{x} + \sqrt{p}\dot{x} + 5x = 0$. The relevant quantity of concern here is $\Delta = p - 4(2)(5) = p - 40$.

$$\Delta \text{ is } \begin{cases} \text{negative if} & p < 40 \implies \text{underdamped} \\ \text{equals 0 if} & p = 40 \implies \text{critically damped} \\ \text{positive if} & p > 40 \implies \text{overdamped} \end{cases}$$

iii. If undamped and forced, the differential equation becomes $2\ddot{x} + 5x = \cos(\omega_f t)$ with circular frequency $\omega_0 = \sqrt{k/m} = \sqrt{5/2}$. For resonance, we want

$$\omega_f = \omega_0 = \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$$

3. [20 pts] One end of a spring with restoring (spring) constant of 2 N/m is hooked to a wall, with a 2 kg mass attached to the other the spring's other end. This harmonic oscillator is designed such that the damping constant is twice the magnitude of the restoring (spring) constant. The oscillator is driven with the forcing function $f(t) = 32 \sin t - 24 \cos t$ Newtons. At time $t = 0$, the mass is pulled 9 m to the right of the equilibrium position and then pushed leftward at 11 m/s.

(a) [3 pts] Write the initial value problem modeling this situation.

(b) [14 pts] Solve the initial value problem. (Hint: all constants that need to be found will be integers)

(c) [3 pts] Find the amplitude of the steady-state solution.

SOLUTION:

(a) We have $m = 2, b = 4, k = 2$. With $x(t)$ describing the displacement of the mass, we have

$$2\ddot{x} + 4\dot{x} + 2x = 32 \sin t - 24 \cos t, \quad x(0) = 9,\ \dot{x} = -11$$

(b) Characteristic equation $2r^2 + 4r + 2 = 2(r^2 + 2r + 1) = 2(r + 1)^2 = 0 \implies r = -1$ with algebraic multiplicity of 2. The solution to the homogeneous equation is thus $x_h(t) = c_1 e^{-t} + c_2 te^{-t}$. Letting $x_p = A \cos t + B \sin t$, substitution into the ODE gives

$$-2A \cos t - 2B \sin t - 4A \sin t + 4B \cos t + 2A \cos t + 2B \sin t = 32 \sin t - 24 \cos t$$

$$-4A \sin t + 4B \cos t = 32 \sin t - 24 \cos t \implies A = -8, B = -6$$

$$x_p = -8 \cos t - 6 \sin t$$

The general solution is then

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-t} + c_2 te^{-t} - 8 \cos t - 6 \sin t$$

to which we apply the initial conditions:

$$x(0) = c_1 - 8 = 9 \implies c_1 = 17$$
$$\dot{x}(t) = -17e^{-t} - c_2 te^{-t} + c_2 e^{-t} + 8 \sin t - 6 \cos t$$
$$\dot{x}(0) = -17 + c_2 - 6 = -11 \implies c_2 = 12$$

giving the solution to the IVP as

$$x(t) = e^{-t}(12t + 17) - 8 \cos t - 6 \sin t$$

(c) The steady-state solution is $x_{ss} = -8 \cos t - 6 \sin t$. It's amplitude is $\sqrt{(-8)^2 + (-6)^2} = 10$.

4. [20 pts] The following problems are not related.

(a) [6 pts] Find the general solution of $y''' - 3y'' + y' + 5y = 0$.

(b) [14 pts] Find the general solution of $y''' - 2y'' = 8 + 90e^{3t}$ using the method of undetermined coefficients.

SOLUTION:

(a) The characteristic equation is $r^3 - 3r^2 + r + 5 = 0$. The Rational Roots Theorem states that the only possible rational roots are $\pm 1, \pm 5$. Using synthetic division we have

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 1 & 5 \\ & & -1 & 4 & -5 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

showing that $r = -1$ is a root. The other roots are obtained using the quadratic formula as $r = 2 \pm i$, giving the general solution of

$$y(t) = c_1 e^{-t} + e^{2t} (c_2 \cos t + c_3 \sin t)$$

(b) Characteristic equation is $r^3 - 2r^2 = r^2(r-2) = 0 \implies r = 2, 0$ with 0 having algebraic multiplicity of 2. Thus

$$y_h(t) = c_1 + c_2 t + c_3 e^{2t}$$

Based on the form of the forcing function, $y_p = A + Be^{3t}$. However, since 1 and t are both solutions to the homogeneous equation we use $y_p = At^2 + Be^{3t}$. Substituting this into the ODE gives

$$27Be^{3t} - 2\left(2A + 9Be^{3t}\right) = 9Be^{3t} - 4A = 8 + 90e^{3t} \implies A = -2, B = 10 \implies y_p = 10e^{3t} - 2t^2$$

Then the general solution is

$$y(t) = y_h + y_p = c_1 + c_2 t + c_3 e^{2t} + 10e^{3t} - 2t^2$$

■

5. [20 pts] Use Laplace transforms to solve the initial value problem $y' - y = 1 + te^t,\ y(0) = 2$

SOLUTION:

$$\begin{aligned}
\mathscr{L}\left\{y' - y\right\} &= \mathscr{L}\left\{1 + te^t\right\} \\
sY(s) - y(0) - Y(s) &= \frac{1}{s} + \frac{1}{(s-1)^2} \\
(s-1)Y(s) &= \frac{1}{s} + \frac{1}{(s-1)^2} + 2 \\
Y(s) &= \frac{1}{s(s-1)} + \frac{1}{(s-1)^3} + \frac{2}{s-1}
\end{aligned}$$

The first term on the right becomes, after partial fraction decomposition,

$$\frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}$$

so that

$$\begin{aligned}
Y(s) &= -\frac{1}{s} + \frac{1}{(s-1)^3} + \frac{3}{s-1} \\
y(t) &= \mathscr{L}^{-1}\left(-\frac{1}{s} + \frac{1}{(s-1)^3} + \frac{3}{s-1}\right) \\
&= \mathscr{L}^{-1}\left(-\frac{1}{s} + \frac{2!}{2(s-1)^{2+1}} + \frac{3}{s-1}\right) \\
&= -1 + \frac{1}{2}t^2 e^t + 3e^t
\end{aligned}$$

■

Short table of Laplace Transforms: $\mathscr{L}\left\{f(t)\right\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a and b can be any real numbers, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2 + b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2 + b^2}$$

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

$$\mathscr{L}\left\{f'(t)\right\} = s\mathscr{L}\left\{f(t)\right\} - f(0) \qquad \mathscr{L}\left\{f''(t)\right\} = s^2\mathscr{L}\left\{f(t)\right\} - sf(0) - f'(0) \qquad \mathscr{L}\left\{tf'(t)\right\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s}$$
