1. [20 pts] Let $y_1 = t^{1/2}$ and $y_2 = t^{-3/2}$ be solutions to the ODE $L(y) = y'' + a_1(t)y' + a_0(t)y = 0, \;\; t > 0$, where $a_1(t)$ and $a_0(t)$ are continuous on $[0, \infty)$.
   (a) [8 pts] Is the set $\{y_1, y_2\}$ a basis for the solution space of the ODE? Justify your answer.
   (b) [12 pts] Find the general solution of $L(y) = 4t^{5/2}$.

2. [20 pts] The following problems are not related.
   (a) [6 pts] Convert the initial value problem $y''' - ty'' + y^2y' - \sqrt{3y} = t^2 + t + 1, \;\; y(0) = -2, \; y'(0) = 0, \; y''(0) = 4$ into an initial value problem for a first order system.
   (b) [14 pts] Consider an harmonic oscillator consisting of a spring attached to a 2 kg object.
      i. [3 pts] A force equal to 10 newtons is required to stretch the spring 2 meters. Find the spring constant, assuming Hooke's Law applies.
      ii. [8 pts] The oscillator is immersed in a medium that offers a damping force equal to $\sqrt{p}$ times the instantaneous velocity. Determine the value(s) of p so that the subsequent motion is (i) underdamped, (ii) critically damped, (iii) overdamped.
      iii. [3 pts] Now suppose that the system is undamped but the motion is forced by $f(t) = \cos(\omega_f t)$. Find the value of ω_f that will put the system into resonance.

3. [20 pts] One end of a spring with restoring (spring) constant of 2 N/m is hooked to a wall, with a 2 kg mass attached to the other the spring's other end. This harmonic oscillator is designed such that the damping constant is twice the magnitude of the restoring (spring) constant. The oscillator is driven with the forcing function $f(t) = 32\sin t - 24\cos t$ Newtons. At time $t = 0$, the mass is pulled 9 m to the right of the equilibrium position and then pushed leftward at 11 m/s.
   (a) [3 pts] Write the initial value problem modeling this situation.
   (b) [14 pts] Solve the initial value problem. (Hint: all constants that need to be found will be integers)
   (c) [3 pts] Find the amplitude of the steady-state solution.

4. [20 pts] The following problems are not related.
   (a) [6 pts] Find the general solution of $y''' - 3y'' + y' + 5y = 0$.
   (b) [14 pts] Find the general solution of $y''' - 2y'' = 8 + 90e^{3t}$ using the method of undetermined coefficients.

5. [20 pts] Use Laplace transforms to solve the initial value problem $y' - y = 1 + te^t, \; y(0) = 2$

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a and b can be any real numbers, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a)$$

$$\mathscr{L}\left\{f'(t)\right\} = s\mathscr{L}\left\{f(t)\right\} - f(0) \qquad \mathscr{L}\left\{f''(t)\right\} = s^2\mathscr{L}\left\{f(t)\right\} - sf(0) - f'(0) \qquad \mathscr{L}\left\{tf'(t)\right\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s}$$