1. [14 pts] In your bluebook, write the word **TRUE** if the statement is always true or write the word **FALSE** if the statement is false. No justification needed and no partial credit given.

   (a) All systems of homogeneous linear algebraic equations are consistent.

   (b) If $\mathbf{A}$ is an $n \times n$ matrix such that $|\mathbf{A}| = 0$, and $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$, then the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ will always have infinitely many solutions.

   (c) The matrix $\begin{bmatrix} 1 & 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is in RREF.

   (d) $\left[(\mathbf{AB})^{-1}\right]^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\right)^{-1}$

   (e) If matrix $\mathbf{A}$ is a 2×6 matrix and $\mathbf{AB}$ is a 2×4 matrix, then $\mathbf{B}$ is a 6×4 matrix.

   (f) For any $n \times n$ matrix $\mathbf{A}$, the following statements are equivalent: $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for any $\vec{\mathbf{b}}$, 0 is an eigenvalue of $\mathbf{A}$, $\mathbf{A}$ is invertible, $\mathbf{A}$ has n linearly independent column vectors, $|\mathbf{A}| \neq 0$.

   (g) A set of 10 vectors in a vector space whose dimension is 9 is always linearly dependent.

SOLUTION:

(a) **TRUE** They all have the trivial solution and perhaps nontrivial solutions as well.

(b) **FALSE** The system may be inconsistent, having no solution.

(c) **FALSE** The 1 in the first row, last column needs to be a zero.

(d) **FALSE** $\left[(\mathbf{AB})^{-1}\right]^{\mathrm{T}} = \left(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\right)^{-1}$

(e) **TRUE** $(2 \times 6)(6 \times 4) \to 2 \times 4$

(f) **FALSE** All statements are equivalent except the 0 eigenvalue.

(g) **TRUE** A set containing more vectors than the dimension of the vector space is necessarily linearly dependent.

■

2. [24 pts] The following problems are not related.

   (a) [6 pts] Let $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ -2 \\ k \end{bmatrix}$. For what value of k is $\vec{\mathbf{w}}$ in Span$\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$? For this value of k, write $\vec{\mathbf{w}}$ as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

   (b) [8 pts] Is the set $\left\{7, 1-t, 1+5t-t^2, \frac{1}{3}t^3+4t\right\}$ a basis for $\mathbb{P}_3$? Explain briefly using the Wronskian, if possible, to make your determination. If using the Wronskian is not possible, answer the question using another method.

   (c) [10 pts] Decide if the following subsets $\mathbb{W}$ of the given vector space $\mathbb{V}$ are subspaces. Assume that the standard operations of vector addition and scalar multiplication apply. Justify the correct answer completely for full credit. A simple yes/no will result in zero points.

      i. $\mathbb{V} = \mathbb{M}_{22}$; $\quad \mathbb{W} = \left\{\mathbf{A} \,\middle|\, \mathbf{A}^{\mathrm{T}} = \mathbf{A}\right\}$

      ii. $\mathbb{V} = \mathcal{C}^2(\mathbb{R})$; $\quad \mathbb{W} = \left\{y(t) \,\middle|\, y'' + 3y' + 4y = e^t\right\}$

SOLUTION:

(a) Need to find k such that for c_1, c_2

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ k \end{bmatrix}$$

This is equivalent to

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ k \end{bmatrix} \quad \text{with augmented matrix} \quad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & -2 \\ 0 & 3 & k \end{array}\right]$$

Converting this into RREF yields

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & -2 \\ 0 & 3 & k \end{array}\right] \xrightarrow{R_2^* = -1R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 3 & k \end{array}\right] \xrightarrow{R_3^* = -3R_2 + R_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 9+k \end{array}\right] \xrightarrow{R_1^* = -2R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 9+k \end{array}\right]$$

$k = -9$ for the system to be consistent. If that is the case, then $c_1 = 7$ and $c_2 = -3$ so that $\vec{\mathbf{w}} = 7\vec{\mathbf{u}} - 3\vec{\mathbf{v}}$

(b)

$$W[7, 1-t, 1+5t-t^2, t^3+4](t) = \begin{vmatrix} 7 & 1-t & 1+5t-t^2 & \frac{1}{3}t^3+4t \\ 0 & -1 & 5-2t & t^2+4 \\ 0 & 0 & -2 & 2t \\ 0 & 0 & 0 & 2 \end{vmatrix} = 28 \neq 0$$

Since the Wronskian is nonzero for all t, the functions are linearly independent. Since the set consists of 4 (the dimension of $\mathbb{P}_3$) linearly independent functions, the set is indeed a basis for $\mathbb{P}_3$.

(c) i. Let $\mathbf{A}$ and $\mathbf{B}$ be vectors in $\mathbb{W}$ with $p, q \in \mathbb{R}$. Then

$$(p\mathbf{A} + q\mathbf{B})^{\mathrm{T}} = (p\mathbf{A})^{\mathrm{T}} + (q\mathbf{B})^{\mathrm{T}} = p\mathbf{A}^{\mathrm{T}} + q\mathbf{B}^{\mathrm{T}} = p\mathbf{A} + q\mathbf{B}$$

showing that $\mathbb{W}$ is closed under linear combinations (and thus under addition and scalar multiplication). Therefore, $\mathbb{W}$ is a subspace of $\mathbf{V}$.

ii. $\vec{\mathbf{0}} \notin \mathbb{W} \implies \mathbb{W}$ is not a subspace.

■

3. [24 pts] The following problems are not related.

(a) [8 pts] Use Cramer's rule to find the value of y in the following system. Simplify your final answer.

$$\begin{aligned} -4x \quad\quad + 2z \quad\quad &= 1 \\ 2x + 2y + 3z \quad\quad &= 0 \\ 3x \quad\quad + \ z + 2w &= -1 \\ -3x \quad\quad + 3z \quad\quad &= 0 \end{aligned}$$

(b) [16 pts] Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

i. Find all eigenvalues of $\mathbf{A}$.

ii. Find a basis for the eigenspace [*i.e.*, find the eigenvector(s)] associated with the repeated eigenvalues (algebraic multiplicity ≥ 2) you found in part i.

<u>SOLUTION:</u>

(a) The determinant of the coefficient matrix is

$$\mathbf{A} = \begin{vmatrix} -4 & 0 & 2 & 0 \\ 2 & 2 & 3 & 0 \\ 3 & 0 & 1 & 2 \\ -3 & 0 & 3 & 0 \end{vmatrix} = 2(-1)^{2+2} \begin{vmatrix} -4 & 2 & 0 \\ 3 & 1 & 2 \\ -3 & 3 & 0 \end{vmatrix} = 2\left[2(-1)^{2+3} \begin{vmatrix} -4 & 2 \\ -3 & 3 \end{vmatrix}\right] = -4\left[(-4)(3) - (-3)(2)\right] = 24$$

and we have

$$\mathbf{A}_2 = \begin{vmatrix} -4 & 1 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 3 & -1 & 1 & 2 \\ -3 & 0 & 3 & 0 \end{vmatrix} = 2(-1)^{3+4} \begin{vmatrix} -4 & 1 & 2 \\ 2 & 0 & 3 \\ -3 & 0 & 3 \end{vmatrix} = -2\left[1(-1)^{1+2} \begin{vmatrix} 2 & 3 \\ -3 & 3 \end{vmatrix}\right] = 2\left[(2)(3) - (-3)(3)\right] = 30$$

Thus, $y = \dfrac{|\mathbf{A}_2|}{|\mathbf{A}|} = \dfrac{30}{24} = \dfrac{5}{4}$.

(b) i.

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-1)^{3+3} \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = -(2+\lambda)\left[(1-\lambda)^2 - 9\right] = 0$$

$$\implies \lambda = -2; \quad (1-\lambda)^2 - 9 = 0 \implies 1 - \lambda = \pm 3 \implies \lambda = 1 \mp 3 = -2, 4$$

So $\lambda = -2, 4$ with -2 having multiplicity 2 (repeated eigenvalue)

ii. Solve the system $[\mathbf{A} - (-2\mathbf{I})]\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

giving $\vec{\mathbf{v}} = \begin{bmatrix} -r \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $r, s \in \mathbb{R}$. Thus a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

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4. [20 pts] The following problems <u>are</u> related.

(a) [5 pts] Assuming that all necessary matrices are invertible (nonsingular), show that if $\mathbf{AB} + \mathbf{I} = \mathbf{A}^2 + \mathbf{B}$, then

$$\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}\left(\mathbf{I} - \mathbf{A}^2\right)$$

(b) [15 pts] Using the result in part (a), calculate $\mathbf{B}$ if $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

SOLUTION:

(a)

$$\begin{aligned} \mathbf{AB} + \mathbf{I} &= \mathbf{A}^2 + \mathbf{B} \\ \mathbf{I} - \mathbf{A}^2 &= \mathbf{IB} - \mathbf{AB} \\ \mathbf{I} - \mathbf{A}^2 &= (\mathbf{I} - \mathbf{A})\,\mathbf{B} \\ (\mathbf{I} - \mathbf{A})^{-1}\left(\mathbf{I} - \mathbf{A}^2\right) &= (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})\,\mathbf{B} \\ (\mathbf{I} - \mathbf{A})^{-1}\left(\mathbf{I} - \mathbf{A}^2\right) &= \mathbf{IB} \\ \mathbf{B} &= (\mathbf{I} - \mathbf{A})^{-1}\left(\mathbf{I} - \mathbf{A}^2\right) \end{aligned}$$

(b)

$$\mathbf{I} - \mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -3 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

$$\mathbf{I} - \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

To find $(\mathbf{I} - \mathbf{A})^{-1}$,

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R_1^* = -R_1 \\ R_2^* = -R_2 \\ R_3^* = -R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

Thus

$$(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}\left(\mathbf{I} - \mathbf{A}^2\right) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 0 & -3 & 0 \\ -2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

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5. [18 pts] Consider the linear system

$$\begin{aligned} 2x_1 + 2x_2 \phantom{{}+2x_3+2x_4} &= 0 \\ x_1 - x_2 + 2x_3 + 2x_4 &= -2 \\ x_1 + 2x_2 - x_3 - x_4 &= 1 \end{aligned}$$

(a) [8 pts] Using Gauss-Jordan elimination, find the RREF of the augmented matrix.

(b) [2 pts] Find a particular solution of the system.

(c) [4 pts] Find the general solution of the associated homogeneous system.

(d) [1 pts] The solution set you found in part (c) is a subspace of $\mathbb{R}^4$. What is the dimension of this solution subspace?

(e) [3 pts] What is the general solution to the original linear system?

SOLUTION:

(a)

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2 & -2 \\ 1 & 2 & -1 & -1 & 1 \end{array}\right] \begin{matrix} R_1^*=\frac{1}{2}R_1 \\ R_3^*=-1R_2+R_3 \end{matrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2 & -2 \\ 0 & 3 & -3 & -3 & 3 \end{array}\right] \begin{matrix} R_2^*=-1R_1+R_2 \\ R_3^*=\frac{1}{3}R_3 \end{matrix} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 1 & -1 & -1 & 1 \end{array}\right] \begin{matrix} R_2 \leftrightarrow R_3 \\ R_3^*=2R_2+R_3 \end{matrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_1^*=-1R_2+R+1 \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{RREF}$$

(b) From the RREF, we have $x_3 = r$ and $x_4 = s$ as free variables with $x_1 = -1 - r - s$ and $x_2 = 1 + r + s$. Choosing $r = s = 0$ a particular solution is

$$\vec{\mathbf{x}}_p = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(c) The general solution to the associated homogeneous system is

$$\vec{\mathbf{x}}_h = \begin{bmatrix} -r-s \\ r+s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad r, s \in \mathbb{R}$$

(d) 2

(e) Using the Nonhomogeneous Principle,

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = r \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad r, s \in \mathbb{R}$$

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