Write on the front of your bluebook a grading key (using the printed lines), your name, your lecture section number and instructor. This exam is worth 100 points and has 5 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted. **Please begin each problem on a new page.**
- You are taking this exam in a proctored and honor code enforced environment. Thus, no calculators, cell phones, or other electronic devices are permitted. You are allowed one 8.5”× 11” crib sheet with writing on one side only.

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1. [14 pts] In your bluebook, write the word **TRUE** if the statement is always true or write the word **FALSE** if the statement is false. No justification needed and no partial credit given.

   (a) All systems of homogeneous linear algebraic equations are consistent.

   (b) If $\mathbf{A}$ is an $n \times n$ matrix such that $|\mathbf{A}| = 0$, and $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$, then the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ will always have infinitely many solutions.

   (c) The matrix $\begin{bmatrix} 1 & 3 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is in RREF.

   (d) $\left[(\mathbf{AB})^{-1}\right]^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\right)^{-1}$

   (e) If matrix $\mathbf{A}$ is a 2×6 matrix and $\mathbf{AB}$ is a 2×4 matrix, then $\mathbf{B}$ is a 6×4 matrix.

   (f) For any $n \times n$ matrix $\mathbf{A}$, the following statements are equivalent: $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for any $\vec{\mathbf{b}}$, 0 is an eigenvalue of $\mathbf{A}$, $\mathbf{A}$ is invertible, $\mathbf{A}$ has n linearly independent column vectors, $|\mathbf{A}| \neq 0$.

   (g) A set of 10 vectors in a vector space whose dimension is 9 is always linearly dependent.

2. [24 pts] The following problems are not related.

   (a) [6 pts] Let $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ -2 \\ k \end{bmatrix}$. For what value of k is $\vec{\mathbf{w}}$ in Span$\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$? For this value of k, write $\vec{\mathbf{w}}$ as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

   (b) [8 pts] Is the set $\left\{7, 1 - t, 1 + 5t - t^2, \frac{1}{3}t^3 + 4t\right\}$ a basis for $\mathbb{P}_3$? Explain briefly using the Wronskian, if possible, to make your determination. If using the Wronskian is not possible, answer the question using another method.

   (c) [10 pts] Decide if the following subsets $\mathbb{W}$ of the given vector space $\mathbb{V}$ are subspaces. Assume that the standard operations of vector addition and scalar multiplication apply. Justify the correct answer completely for full credit. A simple yes/no will result in zero points.

      i. $\mathbb{V} = \mathbb{M}_{22}; \quad \mathbb{W} = \left\{\mathbf{A} \,\middle|\, \mathbf{A}^{\mathrm{T}} = \mathbf{A}\right\}$

      ii. $\mathbb{V} = \mathcal{C}^2(\mathbb{R}); \quad \mathbb{W} = \left\{y(t) \,\middle|\, y'' + 3y' + 4y = e^t\right\}$

3. [24 pts] The following problems are not related.

   (a) [8 pts] Use Cramer's rule to find the value of y in the following system. Simplify your final answer.

$$\begin{aligned} -4x \quad\quad + 2z \quad\quad &= 1 \\ 2x + 2y + 3z \quad\quad &= 0 \\ 3x \quad\quad + \ z + 2w &= -1 \\ -3x \quad\quad + 3z \quad\quad &= 0 \end{aligned}$$

   (b) [16 pts] Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

      i. Find all eigenvalues of $\mathbf{A}$.

      ii. Find a basis for the eigenspace [*i.e.*, find the eigenvector(s)] associated with the repeated eigenvalues (algebraic multiplicity ≥ 2) you found in part i.

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4. [20 pts] The following problems are related.

(a) [5 pts] Assuming that all necessary matrices are invertible (nonsingular), show that if $\mathbf{AB} + \mathbf{I} = \mathbf{A}^2 + \mathbf{B}$, then

$$\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1} \left(\mathbf{I} - \mathbf{A}^2\right)$$

(b) [15 pts] Using the result in part (a), calculate $\mathbf{B}$ if $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

5. [18 pts] Consider the linear system

$$\begin{aligned} 2x_1 + 2x_2 \phantom{{}+2x_3+2x_4} &= 0 \\ x_1 - x_2 + 2x_3 + 2x_4 &= -2 \\ x_1 + 2x_2 - x_3 - x_4 &= 1 \end{aligned}$$

(a) [8 pts] Using Gauss-Jordan elimination, find the RREF of the augmented matrix.

(b) [2 pts] Find a particular solution of the system.

(c) [4 pts] Find the general solution of the associated homogeneous system.

(d) [1 pts] The solution set you found in part (c) is a subspace of $\mathbb{R}^4$. What is the dimension of this solution subspace?

(e) [3 pts] What is the general solution to the original linear system?