1. [12 pts] In your bluebook, write the word **TRUE** if the statement is always true or write the word **FALSE** if the statement is false. No justification needed and no partial credit given.

   (a) The general solution to $w' = -aw + b$, where a and b are real constants is, $w = b/a$.

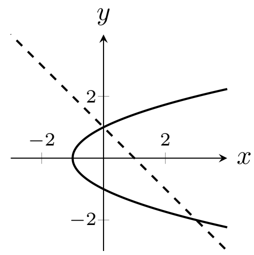
   (b) Picard's theorem guarantees the existence of a unique solution to the initial value problem $y' = \sqrt{t(y-1)},\ y(5) = 1$.

   (c) $y^{(4)} + (\sin t)y = 10y'$ is a fourth order, linear, nonhomogeneous differential equation.

   (d) $e^{\sqrt{2}t}, e^{-\sqrt{2}t}$ and $e^{\sqrt{2}t} - e^{-\sqrt{2}t}$ are all solutions of $y'' - 2y = 0$.

   (e) If c is a real constant, then cy is a always a solution of $y'' + 4y^{0.1} = f(t)$ if y is a solution.

   (f) Consider the system of ODEs $\begin{array}{l} x' = y^2 - x - 1 \\ y' = 1 - x \end{array}$. The following figure correctly shows the v nullcline as the solid curve and the h nullcline as the dashed curve.



SOLUTION:

(a) **FALSE** The Nonhomogeneous Principle applies. The general solution is the sum of the particular solution (b/a) and the solution (Ce^{-at}) to the associated homogeneous ODE.

(b) **FALSE** $f_y = \sqrt{\frac{t}{y-1}}$ is not defined and therefore not continuous at $y = 1$.

(c) **FALSE** Rewrite as $y^{(4)} + (\sin t)y - 10y' = 0$

(d) **TRUE** Substitute the first two into the equation. That the difference of the first two solutions is a solution follows from the Superposition Principle since the ODE is linear and homogeneous.

(e) **FALSE** The ODE is nonlinear.

(f) **FALSE** The h nullcline is the line $x = 1$.

2. [24 pts] Solve the following differential equations using the indicated method. For full credit you must show all steps leading to the correct solution (do not simply plug into a formula to get an answer). Zero credit will be awarded if the stated method is not used.

   (a) [12 pts] $y' - ty^2 = -t$; separation of variables. Leave your answer in implicit form.

   (b) [12 pts] $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 + y^2 - xy}{x^2}$, $x > 0$; use the substitution $z = y/x$. Express your answer as an explicit function of x.

SOLUTION:

(a) Rewrite the ODE as $y' = t(y^2 - 1)$. Then

$$\frac{\mathrm{d}y}{(y+1)(y-1)} = t\,\mathrm{d}t \qquad \text{(partial fractions)}$$

$$\frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right)\mathrm{d}y = \int t\,\mathrm{d}t$$

$$\frac{1}{2}\left(\ln|y-1| - \ln|y+1|\right) = \frac{1}{2}t^2 + C$$

$$\ln\left|\frac{y-1}{y+1}\right| = t^2 + C$$

(b) Rewrite the ODE as $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$. The substitution $z = y/x$ gives $y = xz$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\dfrac{\mathrm{d}z}{\mathrm{d}x} + z$. The original ODE becomes

$$x\frac{\mathrm{d}z}{\mathrm{d}x} + z = 1 - z + z^2 \implies x\frac{\mathrm{d}z}{\mathrm{d}x} = 1 - 2z + z^2 = (1-z)^2$$

which is now separable and we have

$$\int \frac{\mathrm{d}z}{(1-z)^2} = \int \frac{\mathrm{d}x}{x}$$

$$(1-z)^{-1} = \ln|x| + C$$

$$z = 1 - \frac{1}{\ln x + C} \qquad (|x| = x \text{ since } x > 0)$$

$$\frac{y}{x} = 1 - \frac{1}{\ln x + C}$$

$$y = x - \frac{x}{\ln x + C}$$

■

3. [24 pts] Solve the following initial value problems using the indicated method. For full credit you must show all steps leading to the correct solution (do not simply plug into a formula to get an answer). Zero credit will be awarded if the stated method is not used.

(a) [12 pts] $tx' = -x + t^2,\ x(1) = 4/3,\ t > 0$; Euler-Lagrange Two Stage method (variation of parameters)

(b) [12 pts] $y' + \dfrac{e^t}{1+e^t}y = e^{-t},\ y(0) = 2$; integrating factor

SOLUTION:

(a) Rewrite the ODE as $x' + \dfrac{1}{t}x = t$ with associated homogeneous equation $x_h' + \dfrac{1}{t}x_h = 0$. Solve this using separation of variables.

$$\frac{\mathrm{d}x_h}{\mathrm{d}t} = -\frac{1}{t}x_h$$

$$\frac{\mathrm{d}x_h}{x_h} = -\frac{\mathrm{d}t}{t}$$

$$\ln|x_h| = -\ln|t| + \widetilde{C} = \ln t^{-1} + \widetilde{C} \qquad (t > 0 \implies |t| = t)$$

$$|x_h| = e^{\widetilde{C}}t^{-1} \implies x_h = Ct^{-1}$$

Now let $x_p = v(t)t^{-1}$ so that $x_p' = -vt^{-2} + t^{-1}v'$. Substituting this into the nonhomogeneous ODE yields

$$-vt^{-2} + t^{-1}v' + t^{-1}vt^{-1} = t$$

$$v' = t^2$$

$$v(t) = \frac{1}{3}t^3 \implies x_p = \left(\frac{1}{3}t^3\right)t^{-1} = \frac{1}{3}t^2$$

We then have $x = x_h + x_p = Ct^{-1} + \frac{1}{3}t^2$. Applying the initial condition $x(1) = \frac{4}{3} = C + \frac{1}{3} \implies C = 1$ so that

$$x(t) = x_h + x_p = \frac{1}{t} + \frac{1}{3}t^2$$

(b)

$$\int p(t)\,\mathrm{d}t = \int \frac{e^t}{1+e^t}\,\mathrm{d}t \overset{u=1+e^t}{=} \int \frac{\mathrm{d}u}{u} = \ln\left(1+e^t\right) \implies \mu(t) = 1+e^t \qquad \text{(integrating factor)}$$

$$\left(1+e^t\right)y' + e^t y = \left(1+e^t\right)e^{-t}$$

$$\int \left[\left(1+e^t\right)y\right]'\,\mathrm{d}t = \int \left(e^{-t}+1\right)\mathrm{d}t$$

$$(1+e^t)y = -e^{-t} + t + C$$

$$y = \frac{t - e^{-t} + C}{1+e^t} \qquad \text{apply initial condition}$$

$$y(0) = 2 = \frac{-1+C}{2} \implies C = 5$$

$$y = \frac{t - e^{-t} + 5}{e^t + 1}$$

4. [24 pts] The following problems are not related.

(a) [8 pts] For the differential equation $y' = t(y^3 - y^2 - 4y + 4),\ t > 0$, find all equilibrium solutions, if any exist, and determine their stability.

(b) Consider the initial value problem $y' = y - t,\ y(1) = 0$.

i. [8 pts] Draw the isoclines corresponding to slopes of $-2, -1, 0, 1$ and sketch the solution satisfying the IVP.

ii. [8 pts] With a step size of $h = 0.5$, use Euler's method to approximate the solution of the initial value problem at $t = 2$.

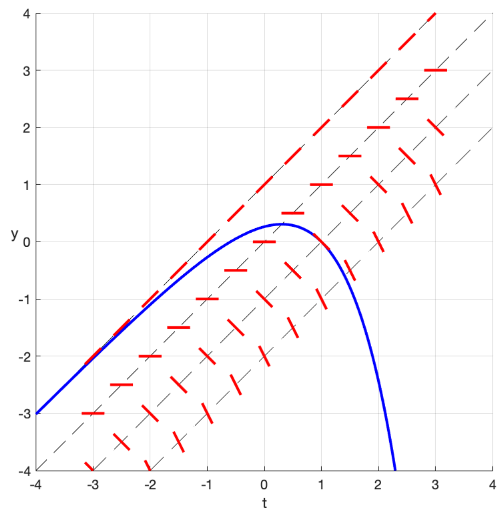
<u>**SOLUTION:**</u>

(a) We can write the ODE as $y' = t(y-1)(y-2)(y+2)$ showing that the equilibrium solutions are $y = -2, y = 1, y = 2$. Noting that $t > 0$ we have

$$\begin{aligned} y > 2:&\quad y' > 0 \\ 1 < y < 2:&\quad y' < 0 \\ -2 < y < 1:&\quad y' > 0 \\ y < -2:&\quad y' < 0 \end{aligned}$$

Thus $y = -2$ and $y = 2$ are unstable and $y = 1$ is stable.

(b) i. Isoclines are the lines $y = t + k$ with $k = -2, -1, 0, 1$.



ii. Using $y_{n+1} = y_n + h(y_n - t_n),\ n = 0, 1$ we have

$$y(1.5) \approx y_1 = y_0 + \tfrac{1}{2}\left(y_0 - t_0\right) = 0 + \tfrac{1}{2}\left(0 - 1\right) = -\tfrac{1}{2}$$

$$y(2.0) \approx y_2 = y_1 + \tfrac{1}{2}\left(y_1 - t_1\right) = -\tfrac{1}{2} + \tfrac{1}{2}\left(-\tfrac{1}{2} - \tfrac{3}{2}\right) = -\tfrac{3}{2}$$

5. [16 pts] The following problems are not related.

(a) [9 pts] A 500-gallon tank is filled with pure water. Brine containing $2(1+\cos t)$ pounds of salt per gallon is pumped into the tank at 5 gallons per minute. The well-mixed solution is pumped out at 10 gallons per minute.

i. [7 pts] Set up, but **DO NOT SOLVE** the initial value problem whose solution will give the amount of salt, $x(t)$, in the tank after t minutes. Write your answer in the form $L(x) = f(t)$ where the linear operator $L(x)$ and $f(t)$ are found from the problem statement.

ii. [2 pts] Without finding the solution, for what values of t will the solution be valid? (Hint: consider what is actually happening physically)

(b) [7 pts] A lake is stocked with 100 fish at the beginning of the year. You have been told by a fisheries biologist that the carrying capacity of the lake is 5000 fish and that the fish population will grow logistically with an initial growth rate of k. Furthermore, the fish are to be harvested from the lake at a rate that is directly proportional (proportionality constant E) to the cube root of the number of fish in the lake at any time. Set up, but **DO NOT SOLVE** the initial value problem whose solution will give the number of fish, $N(t)$, in the lake t months after the beginning of the year.

SOLUTION:

(a) i. Since the tank is initially filled with pure water, the initial condition is $x(0) = 0$.

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \text{Rate in} - \text{Rate out} \\
&= \left[2(1+\cos t)\frac{\text{lb}}{\text{gal}}\right]\left(5\frac{\text{gal}}{\text{min}}\right) - \left(\frac{x}{500+(5-10)t}\frac{\text{lb}}{\text{gal}}\right)\left(10\frac{\text{gal}}{\text{min}}\right) \\
&= 10(1+\cos t) - \frac{2x}{100-t} \\
\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2x}{100-t} &= 10(1+\cos t)
\end{aligned}$$

ii. The volume of brine in the tank at time t is $500+(5-10)t = 500-5t$. The tank will be empty in 100 minutes, after which the differential equation no longer holds. Thus, the solution is valid for $0 \le t < 100$.

(b)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = k\left(1 - \frac{N}{5000}\right)N - E\sqrt[3]{N}, \quad N(0) = 100$$