Write on the front of your bluebook a grading key (using the printed lines), your name, your lecture section number and instructor. This exam is worth 100 points and has 5 questions.

- Submit this exam sheet with your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted. **Please begin each problem on a new page.**
- You are taking this exam in a proctored and honor code enforced environment. Thus, no calculators, cell phones, or other electronic devices are permitted. You are allowed one 8.5”× 11” crib sheet with writing on one side only.

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1. [12 pts] In your bluebook, write the word **TRUE** if the statement is always true or write the word **FALSE** if the statement is false. No justification needed and no partial credit given.

   (a) The general solution to $w' = -aw + b$, where a and b are real constants is, $w = b/a$.

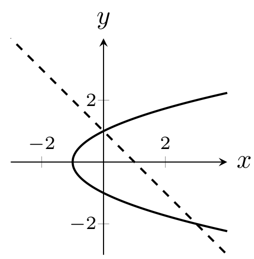
   (b) Picard's theorem guarantees the existence of a unique solution to the initial value problem $y' = \sqrt{t(y-1)},\ y(5) = 1$.

   (c) $y^{(4)} + (\sin t)y = 10y'$ is a fourth order, linear, nonhomogeneous differential equation.

   (d) $e^{\sqrt{2}t}, e^{-\sqrt{2}t}$ and $e^{\sqrt{2}t} - e^{-\sqrt{2}t}$ are all solutions of $y'' - 2y = 0$.

   (e) If c is a real constant, then cy is a always a solution of $y'' + 4y^{0.1} = f(t)$ if y is a solution.

   (f) Consider the system of ODEs $\begin{aligned} x' &= y^2 - x - 1 \\ y' &= 1 - x \end{aligned}$. The following figure correctly shows the v nullcline as the solid curve and the h nullcline as the dashed curve.



2. [24 pts] Solve the following differential equations using the indicated method. For full credit you must show all steps leading to the correct solution (do not simply plug into a formula to get an answer). Zero credit will be awarded if the stated method is not used.

   (a) [12 pts] $y' - ty^2 = -t$; separation of variables. Leave your answer in implicit form.

   (b) [12 pts] $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 + y^2 - xy}{x^2},\ x > 0$; use the substitution $z = y/x$. Express your answer as an explicit function of x.

3. [24 pts] Solve the following initial value problems using the indicated method. For full credit you must show all steps leading to the correct solution (do not simply plug into a formula to get an answer). Zero credit will be awarded if the stated method is not used.

   (a) [12 pts] $tx' = -x + t^2,\ x(1) = 4/3,\ t > 0$; Euler-Lagrange Two Stage method (variation of parameters)

   (b) [12 pts] $y' + \dfrac{e^t}{1 + e^t}y = e^{-t},\ y(0) = 2$; integrating factor

4. [24 pts] The following problems are not related.

   (a) [8 pts] For the differential equation $y' = t(y^3 - y^2 - 4y + 4),\ t > 0$, find all equilibrium solutions, if any exist, and determine their stability.

   (b) Consider the initial value problem $y' = y - t,\ y(1) = 0$.

      i. [8 pts] Draw the isoclines corresponding to slopes of $-2, -1, 0, 1$ and sketch the solution satisfying the IVP.

      ii. [8 pts] With a step size of $h = 0.5$, use Euler's method to approximate the solution of the initial value problem at $t = 2$.

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5. [16 pts] The following problems are not related.

(a) [9 pts] A 500-gallon tank is filled with pure water. Brine containing $2(1+\cos t)$ pounds of salt per gallon is pumped into the tank at 5 gallons per minute. The well-mixed solution is pumped out at 10 gallons per minute.

i. [7 pts] Set up, but **DO NOT SOLVE** the initial value problem whose solution will give the amount of salt, $x(t)$, in the tank after t minutes. Write your answer in the form $L(x) = f(t)$ where the linear operator $L(x)$ and $f(t)$ are found from the problem statement.

ii. [2 pts] Without finding the solution, for what values of t will the solution be valid? (Hint: consider what is actually happening physically)

(b) [7 pts] A lake is stocked with 100 fish at the beginning of the year. You have been told by a fisheries biologist that the carrying capacity of the lake is 5000 fish and that the fish population will grow logistically with an initial growth rate of k. Furthermore, the fish are to be harvested from the lake at a rate that is directly proportional (proportionality constant E) to the cube root of the number of fish in the lake at any time. Set up, but **DO NOT SOLVE** the initial value problem whose solution will give the number of fish, $N(t)$, in the lake t months after the beginning of the year.