APPM 2360: Final Exam
May 6, 2019

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **2 sided**) crib sheet is allowed.

Problem 1: (30 points) The following questions are unrelated.

(a) (12 points) Indicate if the following differential equations are separable or not separable.
  (i) $y' = \frac{t}{y}$
  (ii) $y' = 6e^{yt}$
  (iii) $ty' = 2 + y^3$
  (iv) $y' = \frac{t}{y+1} + \frac{y}{t-1}$

(b) (18 points) Find the general solution of $ty' = y - t^2$ **using the integrating factor method.**

Solution:

(a) (i) Yes
  (ii) No
  (iii) Yes
  (iv) No

(b) First divide through by t and rearrange to be of the form $y' + p(t)y = f(t)$,

$$y' - \frac{1}{t}y = -t.$$

Thus, $\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln(t)} = \frac{1}{t}$. Multiplying by μ to both sides,

$$\frac{d}{dt}\left[\frac{1}{t}y\right] = -1.$$

Integrating both sides,

$$\frac{y}{t} = -t + C.$$

The general solution is then,

$$y(t) = -t^2 + Ct.$$

Problem 2: (50 points; 5 points each) **True/False** (answer True if it is always true, otherwise answer False). No justification is required as there is no partial credit on this question.

(a) If the row reduced echelon form (RREF) of a matrix A is the matrix R and $|R| = 0$, then $|A| = 0$.

(b) It is possible for a system of linear differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ to have exactly 2 equilibrium points.

(c) The solution to the initial value problem given by $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{y}$ and $y(0) = 2$ exists and is unique on an interval around $t = 0$.

(d) The set W of polynomials of degree equal to 2 is a subspace of $\mathbb{P}^2$, the space of all polynomials of degree less than or equal to 2.

(e) In solving $y'' + 4y = \cos(2t)$ using the Method of Undetermined Coefficients, the correct form of the guess for the particular solution is $y_p = A\cos(2t) + B\sin(2t)$

(f) The equation $x'' + x' + 9x = 4\sin(3t)$ has no steady state in the limit $t \to \infty$.

(g) Solutions to the forced and damped oscillator equation $y'' + 2y' + 2y = \mathrm{e}^{-t}\cos(t)$ will decay to zero in the limit $t \to \infty$.

(h) If y_1 and y_2 solve the linear differential equation $L(y) = f(t)$, then $2y_1 - y_2$ also solves $L(y) = f(t)$.

(i) If $\lambda = 2$ is an eigenvalue of an invertible matrix A, with corresponding eigenvector $\mathbf{v}$, then $\lambda = 1/2$ is an eigenvalue of the matrix A^{-1}, also with corresponding eigenvector $\mathbf{v}$.

(j) A mixture with 1g/L sodium is flowed into a tank at a rate of 1L/min. The resulting mixture flows out of the tank at 1L/min. The tank initially contains 10L of pure water. The amount $x(t)$ in grams of sodium in the tank as a function of time (minutes) is described by the initial value problem:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{x}{10} \quad \text{with} \quad x(0) = 0.$$

Solution:

(a) True: in reducing A to R, the determinant is multiplied by -1 when rows are switched, multiplied by a nonzero number when a row is rescaled, and unchanged when a multiple of one row is added to another. Thus, the only way for $|R| = 0$ is if $|A| = 0$.

(b) False: the number of solutions to $A\mathbf{x} = \mathbf{0}$ can only be zero, one, or infinite.

(c) True: $f(t, y) = \frac{1}{y}$ is continuous at $y = 2$ and $\frac{d}{dy}f(t, y) = \frac{-1}{y^2}$ is also continuous at $y = 2$, so Picard's Theorem applies.

(d) False: The zero polynomial is not in the set of polynomials of degree equal to 2. Moreover, it is not closed under addition: $x^2 + 1$ is in W and $-x^2$ is in W, but their sum is 1, which is not in W.

(e) False: since the homogeneous solution is $y_h = c_1\cos(2t) + c_2\sin(2t)$, the correct form of the guess for the particular solution is $y_p = t(A\cos(2t) + B\sin(2t))$

(f) False: The force is periodic and the friction guarantees the absence of the resonance. The steady state is of the form $x_{ss}(t) = A\cos(3t - \delta)$ for some constant A and δ.

(g) True: Even though characteristic roots are $r = -1 \pm i$ and homogeneous solutions are $\mathrm{e}^{-t}\sin(t)$ and $\mathrm{e}^{-t}\cos(t)$ and particular solutions have form $y_p = At\mathrm{e}^{-t}\sin(t) + Bt\mathrm{e}^{-t}\cos(t)$, these decay in the long time limit to zero.

(h) True: By linearity, $L(2y_1 - y_2) = 2L(y_1) - L(y_2) = 2f(t) - f(t) = f(t)$

(i) True: we know $A\mathbf{v} = \mathbf{2v} \to \mathbf{A^{-1}Av} = \mathbf{A^{-1}2v} \to \mathbf{\frac{1}{2}v} = \mathbf{A^{-1}v}$

(j) True: Since dx/dt is in g/min, we need only compute

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1\ \frac{\mathrm{g}}{\mathrm{L}}\right)\left(1\ \frac{\mathrm{L}}{\mathrm{min}}\right) - \frac{x\ \mathrm{g}}{10L}\left(1\ \frac{\mathrm{L}}{\mathrm{min}}\right).$$

Problem 3: (30 points)

(a) (20 points) You place a pot of boiling water ($T(0) = 100°\text{C}$) outside (where it is $T_a = -20°\text{C}$) at noon; by 1pm it is $T(1) = 20°\text{C}$. Using Newton's Law of Cooling

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T_a - T),$$

where time is in hours, determine how many hours it will be until the water freezes (reaches $T = 0°\text{C}$). Simplify your answer as much as possible.

(b) (10 points) Consider the following simple model of the zombie apocalypse:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(1 - y),$$

where y is the fraction of the world population who are zombies.
If initially $y(0) = 0$, what fraction of the world y becomes zombies in the limit $t \to \infty$?
On the other hand if $y(0) = 0.01$, what fraction of the world y becomes zombies as $t \to \infty$?

Solution:

(a)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T_a - T), \quad T(0) = 100.$$

If we write $T = T_a + U$, then $U' = -kU$ and $T(t) = T_a + A\mathrm{e}^{-kt}$. We know $T_a = -20$, so

$$T(0) = -20 + A = 100 \quad A = 120$$

$$T(1) = -20 + 120\mathrm{e}^{-k} = 20 \quad \Rightarrow \quad \mathrm{e}^k = \frac{120}{40} \quad \Rightarrow \quad k = \ln 3.$$

Thus, we expect that the temperature will reach freezing when

$$T(t) = -20 + 120\mathrm{e}^{-\ln 3t} = 0 \quad \Rightarrow \quad \mathrm{e}^{\ln 3t} = \frac{120}{20} \quad \Rightarrow \quad t\ln 3 = \ln 6 \quad \Rightarrow \quad t = \frac{\ln 6}{\ln 3}.$$

(b) $y' = 0$ when $y = 0$, so if $y(0) = 0$, then in the long time limit $y = 0$.
$y' > 0$ when $0 < y < 1$ and $y' = 0$ when $y = 1$, so if $y(0) = 0.01$, then $y \to 1$ in the long time limit (total zombie apocalypse).

Problem 4: (40 points) Consider the following system of differential equations, where we will assume that $x \geq 0$ and $y \geq 0$:

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(2 - x - y)$$

(a) (10 points) Find all nullclines and the three equilibrium points with $x \geq 0$ and $y \geq 0$.

(b) (15 points) Sketch a phase portrait, in the first quadrant ($x \geq 0$ and $y \geq 0$), showing the nullclines, equilibrium points, the direction of the vector field across all the nullclines, and 3 different solution trajectories starting at $(x(0), y(0)) = (2, 3)$; $(x(0), y(0)) = (1, 1/2)$; and $(x(0), y(0)) = (1/4, 1/4)$.

(c) (15 points) Determine whether each of the three equilibria is stable or unstable.

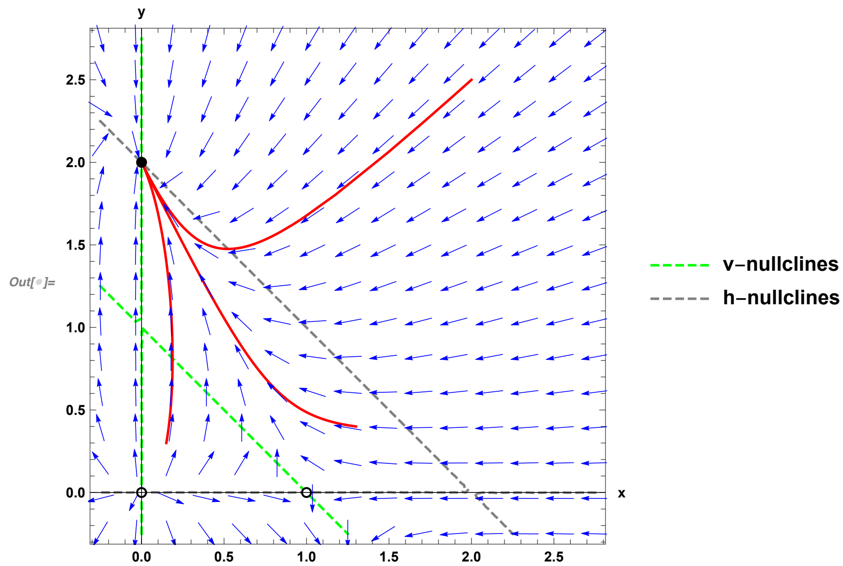
Solution:

(a) The v-nullclines occur when $\frac{dx}{dt} = 0$, that is, when $x = 0$ or when $1 - x - y = 0$, which is the same as $y = -x + 1$.

The h-nullclines occur when $\frac{dy}{dt} = 0$, that is, when $y = 0$ or when $2 - x - y = 0$, which is the same as $y = -x + 2$.

These lines intersect, in the first quadrant, at $(0,0)$, $(0,2)$, $(1,0)$. These are the equilibrium points.

(b) The phase portrait looks like this



(c) $(0,2)$ is stable, $(0,0)$ and $(1,0)$ are both unstable.

Problem 5: (30 points)

(a) (7 points) Determine whether the matrix A is non-singular. If it is non-singular, find its inverse.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(b) (8 points) Find all solutions, $\vec{x}$, to the problem

$$\begin{aligned} 2x_1 - x_2 &= -1 \\ -4x_1 + 2x_2 &= 2. \end{aligned}$$

(c) (8 points) Do the following vectors form a basis in $\mathbb{R}^4$? Explain why or why not.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(d) (7 points) Let W be the subpace of $\mathbb{R}^3$ defined by $W := \{(a, b, c) | a + b + c = 0\}$ find a basis and the dimension of W.

Solution:

(a) Since $|A| = 3 \cdot 6 \cdot (-2) = -36$ the matrix is non-singular with inverse given

$$(A|I) = \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 \end{array}\right) = (I|A^{-1}) \to A^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

No justification is needed for this problem.

(b) Form the augmented matrix and row reduce

$$\left(\begin{array}{cc|c} 2 & -1 & -1 \\ -4 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right).$$

Allowing $x_1 \in \mathbb{R}$ we have $x_2 = 1 + 2x_1$, so for any $x_1 \in \mathbb{R}$ we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 1 + 2x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Alternatively, we could allow $x_2 \in \mathbb{R}$, so $x_1 = x_2/2 - 1/2$, so for any $x_2 \in \mathbb{R}$ we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2/2 - 1/2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.$$

(c) We see that the matrix of vectors is given by the lower-triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and had determinant equal to one. Hence, they are linearly independent. Also, they are four matching the dimension of $\mathbb{R}^4$ and so we conclude that they form a basis.

(d) To obtain a basis for this subspace note that b and c can be free variables, and we require $a = -b - c$, which means that any vector $\mathbf{x} \in W$ is of the form

$$\mathbf{x} = b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = b\mathbf{v}_1 + c\mathbf{v}_2$$

where $\{\mathbf{v}_1, \mathbf{v}_2\}$ denotes one possible basis, and clearly $\dim(W) = 2$.

Problem 6: (40 points)

(a) (10 points) Convert the homogeneous differential equation $y'' + 4y' + 5y = 0$ to a system $\mathbf{x}' = A\mathbf{x}$ of differential equations by setting $x_1 = y$ and $x_2 = y'$, where $A \in \mathbb{R}^{2\times 2}$.

(b) (15 points) Find the eigenvalues $\lambda_{1,2}$ and eigenvectors $\mathbf{v}_{1,2}$ of the matrix

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

(c) (15 points) Write down the general solution $\mathbf{x}$ to $\mathbf{x}' = A\mathbf{x}$ **for A defined in part (b).**

Solution:

(a) Define $x_1 = y$ and $x_2 = y'$, so

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(b) We can compute

$$|\lambda I - A| = \begin{vmatrix} \lambda+1 & 1 \\ -1 & \lambda+1 \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

which has roots $\lambda_\pm = -1 \pm i$. To find the eigenvector associated with λ_+, we must find the nullspace of

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \to \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$$

so $iv_1 + v_2 = 0$ so using $v_1 = 1$, we have $v_2 = -i$, so

$$\mathbf{v}_\pm = \begin{pmatrix} 1 \\ \mp i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

(c) We can now write

$$\mathbf{x} = \mathrm{e}^{-t}\left(c_1\left[\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] + c_2\left[\sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos(t)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]\right).$$

It is also acceptable to write

$$\mathbf{x} = c_1\mathrm{e}^{(-1+i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2\mathrm{e}^{(-1-i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Problem 7: (30 points) Consider the differential equation $y'' + y = f(t)$ with initial conditions $y(0) = 0$, $y'(0) = 0$, and forcing function $f(t) = 2\mathrm{e}^{-t} + \delta(t - c)$.

(a) (8 points) Compute $F(s)$, the Laplace transform of the forcing function $f(t)$ and state the s values for which it is defined. You may use the Table of Laplace transforms below.

(b) (8 points) Solve for the Laplace transform of the solution $Y(s)$ to the initial value problem. Again, you may use the Table of Laplace transforms below.

(c) (14 points) Rearrange $Y(s)$ and invert Laplace transforms to find the solution $y(t)$ to the initial value problem. Simplify the solution as much as possible.

Solution:

(a)

$$F(s) = \frac{2}{s+1} + e^{-cs}, \qquad s > 0,\ c \geq 0.$$

(b) Note that $\mathcal{L}(y'' + y) = (s^2 + 1)Y(s)$, so solving for $Y(s)$,

$$Y(s) = \frac{2}{(s^2+1)(s+1)} + \frac{e^{-cs}}{s^2+1}$$

(c) The first term of the right-hand-side can be expanded by partial fraction decomposition to obtain

$$Y(s) = -\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s+1} + \frac{e^{-cs}}{s^2+1}$$

Using the Laplace transform table and properties of the step function,

$$y(t) = -\cos(t) + \sin(t) + \mathrm{e}^{-t} + \sin(t - c)H(t - c).$$

Table 1. Table of Laplace transforms

$f(t)$	$F(s)$	s domain	$f(t)$	$F(s)$	s domain
1	$\frac{1}{s}$	$s > 0$	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$, n a positive integer
e^{at}	$\frac{1}{s-a}$	$s > a$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$, n a positive integer
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$	$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|$	$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$\delta(t-c)$	e^{-cs}	$c \geq 0$, $s > 0$	$H(t-c)$	$\frac{e^{-sc}}{s}$	$c \geq 0$, $s > 0$
$f'(t)$	$sF(s) - f(0)$	depends on $f(t)$	$f(t-c)H(t-c)$	$e^{-cs}F(s)$	$c \geq 0$, $s > 0$

n^{th} order derivative: $\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$