APPM 2360: Midterm exam 3
April 17, 2019

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **2 sided**) crib sheet is allowed.

Problem 1: (30 points)

(a) (8 points) The numbers $r = 1, -2 + i$ are characteristic roots of the 3rd order homogeneous differential equation of the form $y''' + by'' + cy' + dy = 0$. Find the constant coefficients b, c and d.

(b) (14 points) Solve the IVP

$$y'' - 4y' + 8y = 0, \qquad y(0) = 0,\, y'(0) = 1.$$

(c) (8 points) Find the general solution to the differential equation $y^{(4)} - 2y''' + y'' = 0$.

Solution:

(a) Our third root needs to be the complex conjugate of $-2 + i$. Thus, the three roots are $r = 1, -2 + i, -2 - i$. The characteristic equation associated is then,

$$(r - 1)(r + 2 - i)(r + 2 + i) = r^3 + 3r^2 + r - 5 = 0$$

This gives us the DE

$$y''' + 3y'' + y' - 5 = 0.$$

(b) The characteristic equation is $r^2 - 4r + 8 = 0$. The roots

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)} = 2 \pm 2i$$

Thus, the general solution is

$$y(t) = e^{2t} \left[A \cos(2t) + B \sin(2t)\right]$$

Note that $y(0) = A = 0$ by applying the first initial value. Taking the derivative

$$y'(t) = 2Be^{2t} \left[\cos(2t) + \sin(2t)\right]$$

Applying the second initial value $y'(0) = 2B = 1$. The final solution is then,

$$y(t) = \frac{1}{2} e^{2t} \sin(2t).$$

(Alternatively, the general solution of the form $y(t) = Ce^{2t} \cos(2t - \delta)$ can be used to find $C = 1/2$ and $\delta = \pi/2$.)

(c) The characteristic equation for this DE is

$$r^4 - 2r^3 + r^2 = 0$$

The left hand side can be rewritten,

$$r^2 \left(r^2 - 2r + 1\right) = r^2 (r - 1)^2$$

Thus, the characteristic roots are $r = 0, 0, 1, 1$. The general solution is therefore,

$$y(t) = A + Bt + Ce^t + Dte^t$$

Problem 2: (30 points) Solve the following problems:

(a) (10 pts). Find the general solution to

$$y'' - 2y' + y = 0.$$

(b) (10 pts). Give the general form of the solution to the nonhomogeneous equation according to the method of undetermined coefficients (**do not solve for coefficients**):

$$y'' - 2y' + y = 6 + te^t.$$

(You may use your answer from part (a) of this problem.)

(c) (10 pts). For what values of ω will the following harmonic oscillator system exhibit resonance?

$$x''(t) + \omega^2 x(t) = 3\cos(t).$$

Solution:

(a) The characteristic equation is $r^2 - 2r + 1 = (r-1)^2$. Thus, $r = 1$ is a double root and so the general solution has the form:

$$y(t) = C_1 e^t + C_2 te^t.$$

(b) The RHS of the equation has two terms $y_1 = 6$ and $y_2 = te^t$. Thus, we know that the particular solution will be the sum of two terms. We find the two terms separately. The first term $y_1 = 6$ means that we should look for a constant function A. However, the second term is a solution to the homogeneous equation (and $r = 1$ was a double root) thus the particular solution will have the form $t^2(Bt + C)e^t$. Putting this together we see that

$$y(t) = C_1 e^t + C_2 te^t + A + (Bt^3 + Ct^2)e^t.$$

(c) The homogeneous equation has solution $y_h(t) = A\cos(\omega t) + B\sin(\omega t)$. Looking at the forcing term we know that the particular solution will be of the form $t(A\cos(t) + B\sin(t))$ iff $\omega = 1$. Thus, resonance will happen only when $\omega = 1$.

Problem 3: (30 points) Consider the following differential equation: $y'' - y = e^t$. The solution to $y'' - y = 0$ is $y_h = c_1 y_1 + c_2 y_2 = c_1 e^t + c_2 e^{-t}$.

(a) When looking for a particular solution y_p using variation of parameters, what is the general form of y_p?

(b) Solve for the varying parameters $v_1(t)$ and $v_2(t)$.

(c) Write the specific form of y_p.

(d) Write the general solution y and simplify it as much as possible.

(e) Does the method of undetermined coefficients apply to this differential equation? If so, what is the appropriate form of y_p? Explain. Does this agree with your answer above?

Solution:

(a) $y_p = v_1(t)e^t + v_2(t)e^{-t}$

(b) $W(y_1, y_2) = -2$, so $v_1' = \frac{1}{2} \to v_1(t) = \frac{1}{2}t$ and $v_2' = -\frac{e^{2t}}{2} \to v_2(t) = -\frac{1}{4}e^{2t}$

(c) $y_p = \frac{1}{2}te^t - \frac{1}{4}e^{2t}e^{-t} = \frac{1}{2}te^t - \frac{1}{4}e^t$

(d) $y = y_h + y_p = c_1 e^t + c_2 e^{-t} + \frac{1}{2}te^t$

(e) Yes, the method of undetermined coefficients may be used to solve this problem. The correct form of y_p is Ate^t because the RHS e^t is in y_h. Yes, this agrees with the answer above!

Problem 4: (30 points)

(a) (5 points) Use the **integral definition** (do not use a table) to calculate the Laplace transform $F(s)$ of the function $f(t) = e^{-3t}$. For which s is it defined?

(b) (5 points) Use a table and properties of the Laplace transform to find $\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$.

(c) (20 points) Use the Laplace transform to solve the IVP

$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1, \quad y'(0) = 5$$

Solution:

(a) By definition

$$\begin{aligned}
\mathcal{L}\{e^{-3t}\} &= \int_0^\infty e^{-st}e^{-3t}\,dt \\
&= \lim_{b\to\infty}\int_0^b e^{-(s+3)t}\,dt \\
&= \lim_{b\to\infty} \left.\frac{-e^{-(s+3)t}}{s+3}\right|_0^b \\
&= -\lim_{b\to\infty}\left(\frac{e^{-(s+3)b}}{s+3} - \frac{1}{s+3}\right) \\
&= -\left(0 - \frac{1}{s+3}\right) \\
&= \frac{1}{s+3},
\end{aligned}$$

where we need that $s + 3 > 0$, or $s > -3$, in order for $e^{-(s+3)b} \to 0$ as $b \to \infty$.

(b) Note that

$$\frac{-2s+6}{s^2+4} = -2\cdot\frac{s}{s^2+4} + \frac{6}{2}\cdot\frac{2}{s^2+4}$$

so that

$$\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = -2\cos(2t) + 3\sin(2t)$$

(c) Set $Y(s) = \mathcal{L}\{y(t)\}$ and take the Laplace transform of both sides of the given differential equation:

$$\begin{aligned}
\mathcal{L}\{y'' - 3y' + 2y\} &= \mathcal{L}\{e^{3t}\} \\
\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \frac{1}{s-3} \\
[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) &= \frac{1}{s-3} \\
s^2Y(s) - s - 5 - 3[sY(s) - 1] + 2Y(s) &= \frac{1}{s-3} \\
(s^2 - 3s + 2)Y(s) - s - 2 &= \frac{1}{s-3} \\
(s^2 - 3s + 2)Y(s) &= s + 2 + \frac{1}{s-3} \\
Y(s) &= \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s-3)(s^2 - 3s + 2)} \\
Y(s) &= \frac{s+2}{(s-1)(s-2)} + \frac{1}{(s-3)(s-1)(s-2)}
\end{aligned}$$

Partial fraction decomposition yields

$$Y(s) = \frac{-3}{s-1} + \frac{4}{s-2} + \frac{1}{2(s-3)} + \frac{1}{2(s-1)} - \frac{1}{s-2} = \frac{1}{2(s-3)} + \frac{3}{s-2} - \frac{5}{2(s-1)}$$

so that

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \frac{1}{2}e^{3t} + 3e^{2t} - \frac{5}{2}e^{t}
\end{aligned}$$

Problem 5: (30 points) **True/False** (answer True if it is always true, otherwise answer False). No justification is required as there is no partial credit on this question.

(a) (6 points) For some parameters, the damped oscillator equation $my'' + by' + ky = 0$ has some solutions like $y(t) = te^{-at}$, but for other parameters, it can have solutions like $y(t) = e^{-at}\cos(\omega t)$.

(b) (6 points) For all choices of coefficients a, b, c, particular solutions of the differential equation $ay'' + by' + cy = t^2$ are of the form $y_p(t) = At^2 + Bt + C$.

(c) (6 points) The amplitude of solutions to $y'' + 2y' + 2y = \cos(t)$ grows indefinitely.

(d) (6 points) The Laplace transform $\mathcal{L}$ of $f(t) = 0$ cannot be defined.

(e) (6 points) The initial value problem $y'' + y = e^t$ with $y(0) = 1$ and $y'(0) = 0$ can be solved using method of undetermined coefficients, variation of parameters, or Laplace transforms, and the resulting solutions are all the same.

Solution:

(a) **True.** If $m = 1$, $b = 2$, and $k = 1$ then $y = te^{-t}$ is a solution, and if $m = 1$, $b = 2$, and $k = 2$, then there are solutions of the form $y = e^{-t}\cos(t)$.

(b) **False.** If $a = 1$ and $b = c = 0$, the solution is $t^4/12$.

(c) **False.** Homogeneous solutions of a damped oscillator equation have amplitude that decays. Particular solutions $y_p = (1/5)\cos(t) + (2/5)\sin(t)$ have constant amplitude.

(d) **False.** Compute $\mathcal{L}\{0\} = \int_0^\infty e^{-st} \cdot 0 \, dt = 0$. Also note that be linearity for any $g(t)$, $\mathcal{L}\{g(t)\} = \mathcal{L}\{g(t) + 0\} = \mathcal{L}\{g(t)\} + \mathcal{L}\{0\}$, so subtracting $\mathcal{L}\{g(t)\}$, we have $\mathcal{L}\{0\} = 0$.

(e) **True.** All methods give the solution $y = (1/2)(\cos(t) - \sin(t) + e^t)$.

Short table of Laplace Transforms

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} \quad \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2} \quad \mathcal{L}\{\cosh(bt)\} = \frac{s}{s^2-b^2} \quad \mathcal{L}\{\sinh(bt)\} = \frac{b}{s^2-b^2}$$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$