APPM 2360: Midterm exam 2
March 13, 2019

On the front of your Bluebook write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **2 sided**) crib sheet is allowed.

Problem 1: (30 points). Consider the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & \alpha \end{bmatrix}$$

(a) (6 points) Find the eigenvalues of the matrix A.
(b) (6 points) For which values of α and β does A^{-1} exist?
(c) (12 points) Find the eigenvectors of the matrix A provided all the eigenvalues are distinct.
(d) (6 points) How many linearly independent eigenvectors can be associated with $\alpha = 1$?

Solution:

a Notice that A is upper triangular. This means the eigenvalues are listed down the diagonal. Thus,

$$\lambda = 1, 2, \alpha$$

b A^{-1} exists if $|A| \neq 0$, which means that no eigenvalue can be zero. Thus, the only restriction is $\alpha \neq 0$.

c Start with the eigenvalue $\lambda = 1$:

$$(A - I)\vec{v} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \beta \\ 0 & 0 & \alpha - 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us that $v_1 + v_2 = 0$ and $v_3 = 0$. Then,

$$\vec{v}_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Next $\lambda = 2$

$$(A - 2I)\vec{v} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & \beta \\ 0 & 0 & \alpha - 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first equation and third equation give us $v_2 = v_3 = 0$. Thus,

$$\vec{v}_{\lambda=2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Finally $\lambda = \alpha$

$$(A - \alpha I)\vec{v} = \begin{bmatrix} 2-\alpha & 1 & 0 \\ 0 & 1-\alpha & \beta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We immediately see a free variable from the bottom row. Let $v_3 = t$. The second equation gives us $v_2 = -\frac{\beta}{1-\alpha}v_3$. And the first equation gives us $v_1 = -\frac{1}{2-\alpha}v_2 = \frac{\beta}{(2-\alpha)(1-\alpha)}v_3$. Then when t=1,

$$\vec{v}_{\lambda=\alpha} = \begin{bmatrix} \frac{\beta}{(2-\alpha)(1-\alpha)} \\ -\frac{\beta}{1-\alpha} \\ 1 \end{bmatrix}$$

d When $\alpha = 1$ the eigenvalue $\lambda = 1$ now has multiplicity two and so there could be one or two linearly independent eigenvectors.

Problem 2:(30 points) **True/False** (answer True if it is always true, otherwise answer False). No justification is required as there is no partial credit on this question.

(a) (6 points) The set $S = \{x^3, x^2 + x + 1, x - 1\}$ is a basis for $\mathbb{P}_3$, the vector space of all polynomials of degree less than or equal to 3.

(b) (6 points) The following set of vectors form a basis for $\mathbb{R}^4$: $\begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \\ 0 \end{pmatrix}$

(c) (6 points) The matrix $\begin{bmatrix} 1 & 2 \\ 2 & k \end{bmatrix}$ is invertible for all $k \neq 0$

(d) (6 points) If the reduced-row echelon form (RREF) of a matrix A is $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $|A| = 1$.

(e) (6points) If $|A| = 5$, then $|B^{-1}AB| = 5$

Solution:

a False: This set of 3 elements is linearly independent, but it cannot span $\mathbb{P}_3$, whose dimension is 4.

b True: The matrix $A = \begin{bmatrix} 10 & 5 & 7 & 1 \\ 0 & 6 & 8 & 2 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ has columns given by the set of vectors in question and $|A| = 10 * 6 * 9 * 4 \neq 0$.

c False: The matrix $\begin{bmatrix} 1 & 2 \\ 2 & k \end{bmatrix}$ is invertible for all $k \neq 4$ as $|A| = k - 4$.

d False: We know $|A| \neq 0$, but $|A|$ is not necessarily equal to one.

e True: $|B^{-1}AB| = |B^{-1}| \times |A| \times |B| = \frac{1}{|B|} \times |A| \times |B| = |A| = 5$.

Problem 3: (30 points) Provide clear justification for each of the unrelated problems listed below. **No points will be awarded if there is no justification.**

(a) (9 pts) Are the functions $f_1(t) = \sin(t)$ and $f_2(t) = \sin(2t)$ linearly independent for $t > 0$? Explain why or why not?

(b) (7 pts) Let $\mathbb{V}$ be the vector space of continuous that have derivatives of all orders on $[0, 1]$ and $\mathbb{W} = \{f(t), t \in [0, 1] | f'(t) \geq 0\}$ (the set of functions with a first derivative equal to or greater than zero). Is $\mathbb{W}$ a vector subspace of $\mathbb{V}$?

(c) (7 pts) Let $\mathbb{P}_4$ be the vector space of polynomials in x of order 4 or less and $\mathbb{W} = \{kx^4 + x^2 | k \in \mathbb{R}\}$. Is $\mathbb{W}$ a vector subspace of $\mathbb{P}_4$?

(d) (7 pts) Let $\mathbb{V}$ be the vector space of continuous that have three continuous derivatives on $\mathbb{R}$ and $\mathbb{W} = \{y(t), t \in \mathbb{R} : y''' + \sin(t)y' = 3y\}$. Is $\mathbb{W}$ a vector subspace of $\mathbb{V}$?

Solution:

a Use the Wronskian:

$$\begin{aligned} W[f_1, f_2](t) &= \begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} \sin(t) & \sin(2t) \\ \cos(t) & 2\cos(2t) \end{vmatrix} \\ &= 2\sin(t)\cos(2t) - \cos(t)\sin(2t) \end{aligned}$$

Notice that

$$\begin{aligned} W[f_1, f_2](\pi/4) &= 2\sin(\pi/4)\underbrace{\cos(\pi/2)}_{=0} - \cos(\pi/4)\underbrace{\sin(\pi/2)}_{=1} \\ &= -\frac{1}{\sqrt{2}} \\ &\neq 0 \end{aligned}$$

(there are other values of t you can pick, such as $\pi/2$). Since there is at least one t-value for which the Wronskian isn't zero, we can conclude that f_1 and f_2 are in fact linearly independent.

b No, we simply need to break one of the conditions for the definition of a vector subspace. The main things to check are closure under addition or scalar multiplication. In this case, if we consider the functions $f(t), g(t) \in \mathbb{W}$ and the linear combination $h(t) = a_1 f(t) + a_2 g(t)$ with $a_1, a_2 \in \mathbb{R}$. Then, $h'(t) = a_1 f'(t) + a_2 g'(t)$ and one cannot ensure that $h'(t) \geq 0$ for any a_1 and a_2. Therefore, $h \notin \mathbb{W}$ and so $\mathbb{W}$ is not a vector subspace of $\mathbb{V}$. Note, that it is also not closed under scalar multiplication: if $g \in \mathbb{W}$ then $-g \notin \mathbb{W}$. Moreover, you could also check directly that $\mathbb{W}$ is not a vector space in it's own right, for example, $g(x) = x$ does not have a additive inverse in $\mathbb{W}$.

c No, following the discussion from part (a) consider the polynomials $f(x), g(x) \in \mathbb{W}$ and the linear combination $h(x) = a_1 f(x) + a_2 g(x)$ with $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned} h(x) &= a_1 f(x) + a_2 g(x) \\ &= a_1(kx^4 + x^2) + a_2(kx^4 + x^2) \\ &= k(a_1 + a_2)x^4 + (a_1 + a_2)x^2 \end{aligned}$$

$h(x) \in \mathbb{W}$ only if $a_1 + a_2 = 1$ for the quadratic term. Therefore, $\mathbb{W}$ is not a vector subspace of $\mathbb{V}$ for any value of k. An alternative solution is to realize that $0 \notin \mathbb{W}$ and so $\mathbb{W}$ is not a vector space it is own right.

d Yes. The differential equation is homogeneous and a linear operator. Consider the solutions f, g and the linear combination $h = a_1 f + a_2 g$ with $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned} h''' + \sin(t)h' - 3h &= a_1(f''' + \sin(t)f' - 3f) + a_2(g''' + \sin(t)g' - 3g) \\ &= a_1(0) + a_2(0) = 0 \end{aligned}$$

Problem 4: (30 points) The following problems are not related.

(a) (10 pts) Write the following system of algebraic equations in matrix-vector form.

$$\begin{cases} 2x + 3y + 2z = 5, \\ 2z - y = 0, \\ 2x - 4z = 1. \end{cases}$$

(b) (20 pts) Find all solutions to the following system of algebraic equations using row-operations. Separate the solution out into a sum of the homogeneous and particular solutions.

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Solution:

(a)

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 0 & -1 & 2 \\ 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}.$$

(b) The RREF form of the augmented system is:

$$\begin{pmatrix} 1 & 0 & 2 & | -3 \\ 0 & 1 & -7 & |11 \end{pmatrix}.$$

This lead to the solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 1 \end{pmatrix} t + \begin{pmatrix} -3 \\ 11 \\ 0 \end{pmatrix}$$

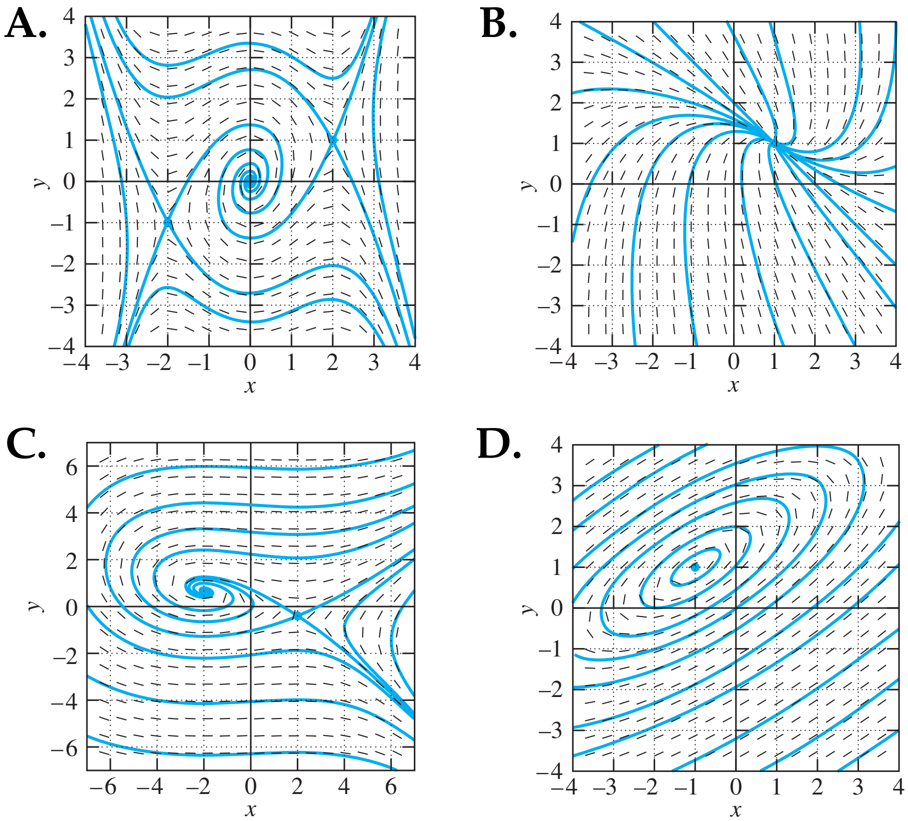
for any real value t.

Problem 5: (30 points) Solve the following unrelated problems involving systems of differential equations.

(a) (20 pts) Match each system to its phase portrait. You do not need to show any work:

$(i)\ \dot{x} = x - y, \quad \dot{y} = x + 3y - 4, \qquad (ii)\ \dot{x} = x - 2y + 3, \ \ \dot{y} = x - y + 2,$

$(iii)\ \dot{x} = 2 - 4x - 15y, \ \ \dot{y} = 4 - x^2 \qquad (iv)\ \dot{x} = x - 2y, \ \ \dot{y} = 4x - x^3.$



(b) (10 pts) Solve for the nullclines and fixed point(s) of the following system.

$$\dot{x} = -x + y, \qquad \dot{y} = 2x - y.$$

Plot and label the nullclines, fixed point(s), and two solution trajectories, and identify the stability of any fixed point(s).

Solution:

a. (i) B; (ii) D; (iii) C; (iv) A

b. x-nullcline: $y = x$; y-nullcline: $y = 2x$; fixed point (0,0) is a saddle, as shown below