Problem 1: (30 points). Consider the following matrix:

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
0 & 1 & \beta \\
0 & 0 & \alpha \\
\end{bmatrix}
\]

(a) (6 points) Find the eigenvalues of the matrix \(A\).
(b) (6 points) For which values of \(\alpha\) and \(\beta\) does \(A^{-1}\) exist?
(c) (12 points) Find the eigenvectors of the matrix \(A\) provided all the eigenvalues are distinct.
(d) (6 points) How many linearly independent eigenvectors can be associated with \(\alpha = 1\)?

Problem 2: (30 points) True/False (answer True if it is always true, otherwise answer False). No justification is required as there is no partial credit on this question.

(a) (6 points) The set \(S = \{x^3, x^2 + x + 1, x - 1\}\) is a basis for \(P_3\), the vector space of all polynomials of degree less than or equal to 3.
(b) (6 points) The following set of vectors form a basis for \(\mathbb{R}^4\):

\[
\begin{pmatrix}
10 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix},
\begin{pmatrix}
5 \\
6 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
7 \\
8 \\
9 \\
0
\end{pmatrix}
\]

(c) (6 points) The matrix \(\begin{bmatrix} 1 & 2 \\ 2 & k \end{bmatrix}\) is invertible for all \(k \neq 0\)

(d) (6 points) If the reduced-row echelon form (RREF) of a matrix \(A\) is

\[
R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

then \(|A| = 1\).

(e) (6 points) If \(|A| = 5\), then \(|B^{-1}AB| = 5\)

Problem 3: (30 points) Provide clear justification for each of the unrelated problems listed below. No points will be awarded if there is no justification.

(a) (9 pts) Are the functions \(f_1(t) = \sin(t)\) and \(f_2(t) = \sin(2t)\) linearly independent for \(t > 0\)? Explain why or why not?

(b) (7 pts) Let \(V\) be the vector space of continuous functions that have derivatives of all orders on \([0, 1]\) and \(W = \{f(t), t \in [0, 1][f'(t) \geq 0}\}\) (the set of functions with a first derivative equal to or greater than zero). Is \(W\) a vector subspace of \(V\)?

(c) (7 pts) Let \(P_4\) be the vector space of polynomials in \(x\) of order 4 or less and \(W = \{kx^4 + x^2 | k \in \mathbb{R}\}\). Is \(W\) a vector subspace of \(P_4\)?

(d) (7 pts) Let \(V\) be the vector space of continuous functions that have three continuous derivatives on \(\mathbb{R}\) and \(W = \{y(t), t \in \mathbb{R} : y''' + \sin(t)y' = 3y\}\). Is \(W\) a vector subspace of \(V\)?
Problem 4: (30 points) The following problems are not related.
(a) (10 pts) Write the following system of algebraic equations in matrix-vector form.
\[
\begin{cases}
2x + 3y + 2z = 5, \\
2z - y = 0, \\
2x - 4z = 1.
\end{cases}
\]
(b) (20 pts) Find all solutions to the following system of algebraic equations using row-oplications. Separate the solution out into a sum of the homogeneous and particular solutions.
\[
\begin{pmatrix}
3 & 1 & -1 \\
1 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
2 \\
-3
\end{pmatrix}.
\]

Problem 5: (30 points) Solve the following unrelated problems involving systems of differential equations.
(a) (20 pts) Match each system to its phase portrait. You do not need to show any work:
\begin{itemize}
    \item[(i)] \( \dot{x} = x - y, \quad \dot{y} = x + 3y - 4, \)
    \item[(ii)] \( \dot{x} = x - 2y + 3, \quad \dot{y} = x - y + 2, \)
    \item[(iii)] \( \dot{x} = 2 - 4x - 15y, \quad \dot{y} = 4 - x^2 \)
    \item[(iv)] \( \dot{x} = x - 2y, \quad \dot{y} = 4x - x^3. \)
\end{itemize}

(b) (10 pts) Solve for the nullclines and fixed point(s) of the following system.
\[
\dot{x} = -x + y, \quad \dot{y} = 2x - y.
\]
Plot and label the nullclines, fixed point(s), and two solution trajectories, and identify the stability of any fixed point(s).