Problem 1: (30 points) Solve the following unrelated problems.

(a) (6 pts) Is the following differential equation separable or not? Justify your answer.

\[ \frac{dy}{dt} = \frac{\sqrt{t}}{y} + 1. \]

(b) (12 pts) Find an explicit solution \( y(t) \) to the following differential equation using the separation of variables technique.

\[ \frac{dy}{dt} = y^2 - 2ty^2. \]

(c) (6 pts) For the function \( y(t) \) below, determine the constant \( C \) using the initial condition.

\[ y(t) = \frac{1}{C + \sin(t)}, \quad y \left( \frac{\pi}{2} \right) = 1. \]

(d) (6 pts) For the following implicit solution, find all values of \( t \) for which \( y \) is undefined.

\[ \sqrt{y^{-1} + 1} = t \]

Solution:

a No, \( \frac{\sqrt{t}}{y} + 1 \) cannot be written in the form \( f(y)g(t) \).

b

\[ \Rightarrow \frac{dy}{dt} = y^2(1 - 2t) \Rightarrow \int \frac{dy}{y^2} = \int (1 - 2t)dt \Rightarrow -y^{-1} = t - t^2 + C \Rightarrow y(t) = \frac{-1}{t - t^2 + C} \]

c

\[ y \left( \frac{\pi}{2} \right) = \frac{1}{C + \sin(\pi/2)} = \frac{1}{C + 1} = 1 \Rightarrow C + 1 = 1 \Rightarrow C = 0; \quad y(t) = \frac{1}{\sin(t)} = \csc(t). \]

d

\[ \sqrt{y^{-1} + 1} = t \Rightarrow y^{-1} + 1 = t^2 \Rightarrow y(t) = \frac{1}{t^2 - 1}, \]

is undefined at \( t = \pm 1 \). Note also that the original equation is undefined for all \( t < 0 \), thus it is undefined for \( t < 0 \) and \( t = 1 \).

Problem 2: (30 points)

(a) (10 pts) A 2000-gallon swimming pool is initially filled with 1000 gallons of pure water. Water containing 10 lb/gal (pounds per gallon) of chlorine flows into the pool at a rate of 6 gallons per minute. The well-mixed chlorine solution is pumped out at a rate of 4 gallons per minute. Let \( x(t) \) denote the amount of chlorine in the pool at any time \( t > 0 \) (before the pool overflows). Write down an initial value problem that \( x(t) \) must satisfy. State the times for which the initial value problem is valid.

(b) (10 pts) Solve the IVP you found in part (a)
(c) (10 pts) Beer containing 10% alcohol is pumped into a tank that initially contains 500 gallons of beer at 5% alcohol. The rate at which the beer is pumped is 3 gallons per minute. The liquid is thoroughly mixed before it is pumped out at the same rate. Write down the initial value problem for the amount of alcohol in the tank at anytime $t$. How much alcohol is in the tank as $t \to \infty$?

Solution:

Part (a)

a Governing DE: $\frac{dx}{dt} = \text{FLOW IN} - \text{FLOW OUT}$

b FLOW IN = (10)(6)

c FLOW OUT = $\frac{x(t)}{1000+2t}$ (4)

d $x(0) = 0$.

e Answer: $\frac{dx}{dt} = 60 - \frac{4x(t)}{1000+2t}, x(0) = 0$

valid for $0 \leq t < 500 \text{ min}$

Part (b)

a Solution 1: Integrating Factor Method:

Solve $u' = \frac{4}{1000+2t} u$ using separation of variables to get the integrating factor $u(t) = (1000 + 2t)^2$.

Use the integrating factor to solve $(ux)' = 60 \to (1000 + 2t)^2x = \int 60(1000 + 2t)^2 dt \to x(t) = 10(1000 + 2t) + \frac{C}{(1000+2t)^2}$.

Plugging-in the initial condition to solve for $C$ gives $C = -10^{10}$, so $x(t) = 10(1000 + 2t) - \frac{10,000,000,000}{(1000+2t)^2}$.

b Solution 2: Variation of Parameters:

Solve $x' + \frac{4}{1000+2t} x = 0$ using separation of variables to get $x_h = \frac{C}{(1000+2t)^2}$.

Then plug-in $x_p = v(t)(1000 + 2t)^{-2}$ to $x' + \frac{4}{1000+2t} x = 60$ to get $v'(t) = 60(1000 + 2t)^2$

Solve for $v(t)$ to get $v(t) = 10(1000 + 2t)^3$

Then $x_p = v(t)(1000 + 2t)^{-2} = 10(1000 + 2t)$

So $x(t) = x_h + x_p = \frac{C}{(1000+2t)^2} + 10(1000 + 2t)$

Plugging-in the initial condition to solve for $C$ gives $C = -10^{10}$, so $x(t) = 10(1000 + 2t) - \frac{10,000,000,000}{(1000+2t)^2}$.

Part (c)

a IVP $\frac{dx}{dt} = 3(0.1) - 3 \frac{x(t)}{500} = 0.3 - \frac{3x}{500}, x(0) = 500 * (0.05) = 25$.

b As $t \to \infty$ we have that $x(t) = (0.1)500 = 50$. Alternatively, students can realize that it is just 10 percent of 500.

Problem 3: (30 points) Use the integrating factor method to solve the differential equation for $t > 0$: $y' - y = \exp(-t/3)$, where $y(0) = 1$.

(a) (10 pts) Clearly determine the integrating factor.

(b) (5 pts) Multiply the original differential equation by your integrating factor from part (a), and clearly show that the left-hand side of the resulting equation is now the derivative of a product.

(c) (10 pts) Integrate the resulting equation from part (b) and use the initial condition to evaluate any constants.

(d) (5 pts) Write clearly the final solution

Solution:

a The integrating factor is $\rho(t) = \exp(\int (-1) dt) = e^{-t}$.

b Multiplying the original differential equation by the integrating factor results in

$e^{-t}y' - e^{-t}y = e^{-4t/3}$.
The left-hand side is now an exact derivative, specifically \( \frac{d}{dt}(e^{-t}y) \). We have now converted the original differential equation to

\[
\frac{d}{dt}(e^{-t}y) = e^{-4t/3}.
\]

c Integrating yields

\[
e^{-t}y = -(3/4)e^{-4t/3} + C.
\]

Invoking the initial condition shows that \( C = 7/4 \).

d Finally we get that \( y(t) = (-3/4) \exp(-t/3) + (7/4)e^t \).

**Problem 4:** (30 points, 6points each) **True/False** (answer True if it is always true otherwise answer False). No justification is required as there is no partial credit on this question.

(a) The IVP \( y' = \frac{y^3}{t(y-1)} \) with \( y(0) = 2 \) is guaranteed to have a unique solution.  
(b) \( (\frac{dx}{dt})^2 + x = 1 \) is a second order, linear, nonhomogeneous, differential equation.  
(c) A solution for the differential equation \( y'' + y = 2 \) is \( y_1 \). Therefore, \( y_2 = 2y_1 \) is also a solution.  
(d) If the uniqueness condition of Picard’s theorem is not satisfied for an IVP, the solution to that IVP could be still unique.  
(e) An equilibrium for the differential equation \( y' = y^2 - \sqrt{|y|} \) is \( y = -1 \). This equilibrium is unstable.

**Solution:**

(a) **False**: Picard’s theorem does not guarantee existence at \( t = 0 \). Then, uniqueness cannot be guaranteed.  
(b) **False**: A linear algebraic equation can only have variables of power 1.  
(c) **False**: \( y_1 \) is a solution of a nonhomogeneous linear differential equation, \( L(y_1) = 2 \). Therefore, \( L(y_2) = L(2y_1) = 2L(y_1) = 4 \neq 2 \).  
(d) **True**: Picard’s theorem provides sufficient conditions.  
(e) **False**: \( y = -1 \) is a stable equilibrium.

**Problem 5:** (30 points) This short-answer question does not require you to show your work and there is no partial credit. The following parts are unrelated:

(a) (6 pts) Use Euler’s method with a stepsize of \( h = 1/2 \) to approximate the solution to \( \frac{dy}{dt} = y(t+1) \) with initial condition \( y(0) = 2 \), at \( t = 1/2 \).

(b) (6 pts) After 12 hours the size of a bacterial colony is one and a half times its initial size. What is the doubling time for the growth of these bacteria? Do not simplify the answer.

(c) (3pts each) Match the following differential equations to their corresponding direction fields:

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Direction Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ( y' = ty )</td>
<td></td>
</tr>
<tr>
<td>(ii) ( y' = 1 + y )</td>
<td></td>
</tr>
<tr>
<td>(iii) ( y' = \cos(\pi y) )</td>
<td></td>
</tr>
<tr>
<td>(iv) ( y' = \frac{y}{t} )</td>
<td></td>
</tr>
<tr>
<td>(v) ( y' = y(1-y) )</td>
<td></td>
</tr>
<tr>
<td>(vi) ( y' = -y^2 )</td>
<td></td>
</tr>
</tbody>
</table>

Solution:
Solution:

(a) \( y(1/2) \approx y(0) + h \cdot y'(0, 1) \rightarrow y(1/2) \approx 2 + (1/2) \cdot (2 \cdot (0 + 1)) = 3 \)

(b) The size of the colony \( y \) grows with time \( t \) as \( y(t) = y(0)e^{kt} \) where \( k = \ln 2/T \) in terms of the doubling time \( T \). We have \( y(t) = 1.5y(0) \) for the time \( t = 12 \) hrs. Thus, \( 1.5 = e^{12\ln 2/T} \) and so the doubling time

\[
T = 12 \cdot \frac{\ln 2}{\ln 1.5} \text{ hours.}
\]

(c) i) E, ii) C, iii) F, iv) B, v) D, vi) A